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2000 J. Phys. A: Math. Gen. 33 6279

(http://iopscience.iop.org/0305-4470/33/36/302)

J. Phys. A: Math. Gen. **33** (2000) 6279–6309. Printed in the UK PII: S0305-4470(00)12076-1

Towards a cladistics of double Yangians and elliptic algebras*

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Received 21 February 2000

Abstract. A self-contained description of algebraic structures, obtained by combinations of various limit procedures applied to vertex and face $sl(2)$ elliptic quantum affine algebras, is given. New double Yangian structures of dynamical type are defined. Connections between these structures are established. A number of them take the form of twist-like actions. These are conjectured to be evaluations of universal twists.

Contents

* Cladistics: a system of biological taxonomy that defines taxa uniquely by shared characteristics not found in ancestral groups and uses inferred evolutionary relationships to arrange taxa in a branching hierarchy such that all members of a given taxon have the same ancestors.
Cladisme: n. masc. biol. Méthode dont le but est de produire une classification fondée sur le degré de parenté philogénétique des espèces vivantes, plutôt que sur leur aspect extérieur.

1. Introduction

The study of elliptic quantum algebras, defined with the help of elliptic R-matrices, has yielded a number of algebraic structures relevant to certain integrable systems in quantum mechanics and statistical mechanics (noticeably the *XYZ* model [1], RSOS models [2, 3] and sine–Gordon theory [4, 5]). More recently, the definition and construction of some scaling limits has led to the notion of deformed double Yangian algebras. We will investigate and develop here in great detail the occurrence of these and other limit algebraic structures and the pattern of connection between them, in the simplest case of an underlying $sl(2)$ algebra.

Two classes of elliptic solutions to the Yang–Baxter equation have been identified, respectively associated with the vertex statistical models [6, 7] and the face-type statistical models [2, 8, 9]. The vertex elliptic R-matrix for $sl(2)$ was first used by Sklyanin [10] to construct a two-parameter deformation of the enveloping algebra $\mathcal{U}(sl(2))$. The central extension of this structure was proposed in [11] for $sl(2)$, and later extended to $\mathcal{A}_{q,p}(\widehat{sl(N)_c})$ in [12]. Its connection to q-deformed Virasoro and $\mathcal{W}_N$ algebras [13–15] was established in [16, 17].

The face-type R-matrices, depending on the extra parameters λ belonging to the dual of the Cartan algebra in the underlying algebra, were first used by Felder [18] to define the algebra $\mathcal{B}_{q,p,\lambda}(\widehat{sl(2)_c})$ in the R-matrix approach. Enriquez and Felder [19] and Konno [3] introduced a current representation, although differences arise in the treatment of the central extension. A slightly different structure, also based upon face-type R-matrices but incorporating extra, Heisenberg algebra generators, was introduced as $\mathcal{U}_{q,p}(sl(2))$ [3, 20]. This structure is relevant to the resolution of the quantum Calogero–Moser and Ruijsenaar–Schneider models [21–23]. Another dynamical elliptic algebra, denoted as $\mathcal{A}_{q,p;\pi}(\widehat{sl(2)_c})$, was also defined and studied in [24]. It was then interpreted, at the level of representation, as a twist of $\mathcal{A}_{q,p}(\widehat{sl(2)_c})$.

Particular limits of the $\mathcal{A}_{q,p}$-type algebras were subsequently defined and compared with previously known structures. The limit $p \to 0$ together with the renormalization of the generators by suitable powers of p *before* taking the limit, leads to the quantum algebra $\mathcal{U}_q(\widehat{sl(2)_c})$ such as presented in [15, 25]. It differs from the presentation in [26] by a scalar factor in the R-matrix. The scaling limit of the algebra $\mathcal{A}_{q,p;\pi}(\widehat{sl(2)_c})$ was also defined in [24].

A second limit was considered in [4, 27] (R-matrix formulation) and [28] (current algebra formulation). It is defined by taking $p = q^{2r}$ (elliptic nome) and $z = q^{\mathrm{i}\beta/\pi}$ (spectral parameter) with $q \to 1$. This algebra, denoted as $\mathcal{A}_{\hbar,\eta}(\widehat{sl(2)_c})$, where $\eta \equiv \frac{1}{r}$ and $q \simeq \mathrm{e}^{\epsilon\hbar}$ with $\epsilon \to 0$, is relevant to the study of the *XXZ* model in its gapless regime [27]. It admits a further limit $r \to \infty$ ($\eta \to 0$) where its R-matrix becomes identical to the R-matrix defining the double Yangian $\mathcal{DY}(sl(2))_c$ (centrally extended), defined in [29] (Yangian double), [30] (central extension); alternative versions with a different normalization are given in [31] (for $sl(2)$) and [32] (for $sl(N)$). This difference in the normalization factors of the R-matrix, crucial in confronting the centrally extended versions, is the exact counterpart of the difference between the presentation of $\mathcal{U}_q(\widehat{sl(2)_c})$ in [25, 26].

One must, however, be careful in this identification in terms of R-matrix structure since the generating functionals (Lax matrices) of these algebras admit different interpretations in terms of modes (generators of the enveloping algebra). In the context of $\mathcal{A}_{\hbar,0}(\widehat{sl(2)_c})$ the expansion is done in terms of continuous-index Fourier modes of the spectral parameter (see [4, 28]); in the context of $\mathcal{DY}(sl(2))_c$ the expansion is done in terms of powers of the spectral parameter (see [29, 30, 32]).

It was shown recently that both vertex algebras $\mathcal{A}_{q,p}(\widehat{sl(N)_c})$ and face-type algebras $\mathcal{B}_{q,\lambda}(\widehat{sl(N)_c})$ were, in fact, Drinfel'd twists [33] of the quantum group $\mathcal{U}_q(\widehat{sl(N)_c})$. Originating with the proposition of [22] on face-type algebras, the construction of the twist operators was undertaken in both cases by Frønsdal [34, 35] and finally achieved at the level of formal universal twists in [12, 36]. In [36], the universal twist is obtained by solving a linear equation introduced in [37], this equation playing a fundamental rôle for complex continuation of $6j$ symbols. Moreover, in the case of finite (super)algebras, the convergence of the infinite products defining the twists was also proved in [36]. This led to a formal construction of universal R-matrices for the elliptic algebras $\mathcal{A}_{q,p}$ and $\mathcal{B}_{q,\lambda}$, of which the BB and ABF 4×4 matrices are, respectively, (spin-$\frac{1}{2}$) evaluation representations.

2. General settings

Our strategy is to combine in as many patterns as possible the different limit procedures introduced previously in the literature; to apply them to cases not already considered, in particular, the face-type algebras $\mathcal{B}_{q,p,\lambda}(\widehat{sl(2)_c})$; and thus to achieve as large as possible a self-contained network of algebraic structures extending from the elliptic quantum affine algebras to the affine Lie algebra $\mathcal{U}(\widehat{sl(2)_c})$.

Before summarizing our investigations, we must first of all define precisely the concepts which we will use throughout this paper, so that no ambiguity arises in our statements.

We shall deal with formal algebraic structures defined by R-matrix exchange relations between formal 2×2 matrix-valued generating functionals denoted by Lax operators, using the well known *RLL* formalism [38]. Explicit R-matrices are interpreted here as evaluation representations of universal objects whenever they are known to exist, or conjectural universal objects when not. We shall not give any precise definition of the individual generators themselves, i.e. the specific expression of the individual generators in terms of spectral parameter-dependent Lax operators. These definitions would eventually give rise to the fully explicit algebraic structure. For instance, we shall not distinguish here between the double Yangian $\mathcal{D}Y(sl(2))_c$ and the scaled algebra $\mathcal{A}_{\hbar,0}(\widehat{sl(2)_c})$. Definition of, and identification between algebraic structures will therefore be understood at the sole level of their R-matrix presentation, except in explicitly specified cases where we are able to state relations between the full (generator-described) exchange structures, or even the Hopf or quasi-Hopf algebraic structures. We consider that the existence of such relations is in any case an indication that similar connections exist at the level of universal algebras, to be explicitly formulated once the explicit algebra generators are defined.

Similarly we shall manipulate R-matrices at the level of their evaluation representation of spin-$\frac{1}{2}$ (4×4 matrices). Only when shall we use the term 'universal', will it mean the abstract algebraic object known as the universal R-matrix. The same will apply to twist operators connecting (quasi-)Hopf algebraic structures [33], and the R-matrices of the algebras. We recall that a twist operator F lives in the square $\mathcal{A}^{\otimes 2}$ of an algebraic structure; it connects two coproducts in $\mathcal{A}$ as $\Delta_F(\cdot) = F\Delta(\cdot)F^{-1}$, and two universal R-matrices as $R_F = F^{\pi}RF^{-1}$. Its evaluation representation acts similarly on the evaluation representation of the universal R-matrices:

$$R_{12}^F = F_{21}R_{12}F_{12}^{-1}. \tag{2.1}$$

As in the previous case of identifications of algebras, we conjecture that the occurrence of a relation of this form at the level of evaluated R-matrices is an indication that a similar relation exists at the level of universal algebras. We shall therefore denote any such relation between evaluated R-matrices as a 'twist-like action' (TLA) between two algebraic structures, respectively characterized by R and R^F, even when we do not have explicit proof that a universal twist exists between the universal R-matrices, or the respective coproduct structures.

A connection of the form (2.1) where F will not depend on any parameter (spectral (z or β, elliptic (p or r) or dynamical (w or s)) will be termed a 'rigid twist action'.

3. Synopsis

The remainder of our paper is divided into two parts, describing structures and limits (sections 4 and 5) and then twist operations (sections 6 and 7), followed by conclusions and appendices.

In sections 4 and 5 (structures and limits) we shall describe the limit procedures whereby the number of parameters in the R-matrix description of the algebra (hence including the spectral parameter) is decreased, starting from either $\mathcal{A}_{q,p}(\widehat{sl(2)_c})$ or $\mathcal{B}_{q,p,\lambda}(\widehat{sl(2)_c})$; we shall define the limit algebraic structures in both cases. These limit procedures may go in three (for $\mathcal{A}_{q,p}(\widehat{sl(2)_c})$) or four (for $\mathcal{B}_{q,p,\lambda}(\widehat{sl(2)_c})$) directions:

- *non-elliptic limit:* one sends p to 0;
- *scaling limit:* one sends q to 1, with $p = q^{2r}$, $z = q^{i\beta/\pi}$ ($z = q^{2i\beta/\pi}$ and $w = q^{2s}$ in the face case, where w is related to λ, see below);
- *factorization:* one 'eliminates' the spectral parameter by a Sklyanin-type factorization. At the level of the universal algebra this corresponds to a degeneracy homomorphism (see [28]). This procedure is only known for vertex algebras at this point. Finite face-type algebras, however, are known and shall be considered here, albeit without an established connection with the affine structures;
- *non-dynamical limit:* in the face case the dynamical parameter λ can also be eliminated by a procedure which we shall detail in the main body of the text.

These limit procedures, and combinations thereof, lead to the set of objects described by figure 1. Already known structures are of course present in the diagram: $\mathcal{B}_{q,p,\lambda}(\widehat{sl(2)_c})$ is the face elliptic, centrally extended algebra; $\mathcal{A}_{q,p}(\widehat{sl(2)_c})$ is the vertex elliptic, centrally extended algebra; $\mathcal{U}_q^F(\widehat{sl(2)_c})$ and $\mathcal{U}_q^V(\widehat{sl(2)_c})$ are two presentations [12, 39] of the quantum group $\mathcal{U}_q(\widehat{sl(2)_c})$ [25] connected by a conjugation and a twist-like action; $\mathcal{DY}_r^{V8}(\widehat{sl(2)_c})$ is the deformed double Yangian algebra $\mathcal{A}_{\hbar,\eta}(\widehat{sl(2)_c})$ in [28] with $\hbar = 1$ and $\eta = 1/r$; $\mathcal{DY}_r^{V6}(\widehat{sl(2)_c})$ is the deformed double Yangian algebra defined in [4], connected to the previous one by a rigid twist; $\mathcal{DY}(sl(2))_c$ is the double Yangian defined in [29, 30]; $\mathcal{U}_q(sl(2))$ is the q-deformed $sl(2)$ algebra; $\mathcal{S}_{q,p}(sl(2))$ is Sklyanin's elliptic 'degenerate' algebra, and $\mathcal{U}_r(sl(2))$ is the 'degenerate' trigonometric algebra identified with $\mathcal{U}_q(sl(2))$ by $q = e^{i\pi/r}$.

New algebraic structures also appear in this diagram, mostly due to the systematic application of the limit procedures to the face algebra $\mathcal{B}_{q,p,\lambda}(\widehat{sl(2)_c})$: $\mathcal{DY}_{r,s}(sl(2))_c$ is the scaling limit of $\mathcal{B}_{q,p,\lambda}(\widehat{sl(2)_c})$; $\mathcal{DY}_{r,s}^{-\infty}(sl(2))_c$ is its $s \ll 0$ limit where the periodic behaviour in s is nevertheless retained; $\mathcal{DY}_s(sl(2))_c$ is a dynamical deformation of the double Yangian; $\mathcal{U}_{q,\lambda}(\widehat{sl(2)_c})$ and $\mathcal{U}_{q,\lambda}^{\Gamma}(\widehat{sl(2)_c})$ are dynamical deformations of $\mathcal{U}_q(\widehat{sl(2)_c})$, respectively proportional to $\mathcal{DY}_{r,s}^{-\infty}(sl(2))_c$ and $\mathcal{DY}_{r,s}(sl(2))_c$ through a suitable redefinition of the parameters; $\mathcal{DY}_r^F(sl(2))_c$ is an 'elliptic' non-dynamical deformation of the double Yangian, connected to $\mathcal{DY}_r^V(sl(2))_c$ by a twist-like action and proportional to $\mathcal{U}_q(\widehat{sl(2)_c})$ by the same redefinition of the parameters. Finally, $\mathcal{U}_s(sl(2))$ and $\mathcal{B}_{q,\lambda}(sl(2))$ are dynamical deformations of the factorized structures à la Sklyanin, although they themselves are not yet understood as

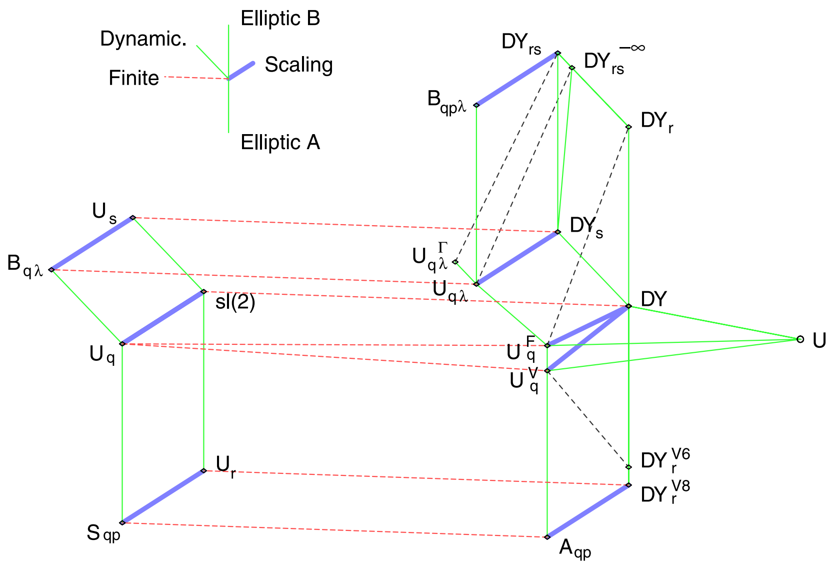

Figure 1. *R*-matrix network.

originating from such a factorization. In addition, we also compare the structures resulting from $\mathcal{B}_{q,p,\lambda}(\widehat{sl(2)_c})$ and the structures derived [24] in the analysis of $\mathcal{A}_{q,p;\pi}(\widehat{sl(2)_c})$. These structures are in, in fact, connected by a TLA which we shall describe.

In order to avoid tedious repetitions in the body of the text, we state immediately that *all* these new *R*-matrices have been explicitly checked to obey the Yang–Baxter equations (A.1) and (A.2) or dynamical Yang–Baxter equations (A.3) and (A.4). Such checks are indeed required since the computational procedures which yield these *R*-matrices may entail regularizations of infinite products. This fact, in turn, potentially invalidates a direct application of these computational procedures to the Yang–Baxter equation originally satisfied by the elliptic *R*-matrices.

In sections 6 and 7 (twist operations) we describe the connections which implement the *addition* of supplementary parameters. To be precise:

- implementation of the elliptic nome p (or r);
- implementation of the dynamical parameter w (or s);
- implementation of the quantum parameter q along the scaling limit connections.

Two types of twist-like actions appear.

(a) TLA explicitly proved to be evaluation of universal twists, represented in the figures 5–7 by a triple arrow. Most of them have been established previously in the literature, particularly in [12, 34, 35, 40].

(b) TLA conjectured to be evaluations of universal twists, represented on the figures 5–7 by a double arrow. All these objects are new. They are either deduced from previously known ones by limit procedures or combinations; or explicitly computed from scratch.

We conclude with some indications on further possible investigations.

Finally, one appendix recalls the different forms of Yang–Baxter equations together with the definitions of twists and cocycle relations. The second appendix collects a few mathematical definitions and formulae related to elliptic functions.

Structures and limits

4. Vertex-type algebras

We will start from the elliptic algebra $\mathcal{A}_{q,p}(\widehat{sl(2)_c})$ and take the above-described different limits to obtain various quantum algebras and deformed double Yangians.

4.1. Elliptic algebra $\mathcal{A}_{q,p}(\widehat{sl(2)_c})$

Let us consider the following R-matrix [6, 11]:

$$R(z,q,p) = \frac{\tau(q^{1/2}z^{-1})}{\mu(z)} \begin{pmatrix} a(z) & 0 & 0 & d(z) \\ 0 & b(z) & c(z) & 0 \\ 0 & c(z) & b(z) & 0 \\ d(z) & 0 & 0 & a(z) \end{pmatrix} \tag{4.1}$$

where

$$a(z) = z^{-1}\frac{\Theta_{p^2}(q^2z^2)\Theta_{p^2}(pq^2)}{\Theta_{p^2}(pq^2z^2)\Theta_{p^2}(q^2)} \tag{4.2}$$

$$b(z) = qz^{-1}\frac{\Theta_{p^2}(z^2)\Theta_{p^2}(pq^2)}{\Theta_{p^2}(pz^2)\Theta_{p^2}(q^2)} \tag{4.3}$$

$$c(z) = 1 \tag{4.4}$$

$$d(z) = -p^{1/2}q^{-1}z^{-2}\frac{\Theta_{p^2}(z^2)\Theta_{p^2}(q^2z^2)}{\Theta_{p^2}(pz^2)\Theta_{p^2}(pq^2z^2)}. \tag{4.5}$$

The normalization factors are

$$\frac{1}{\mu(z)} = \frac{1}{\kappa(z^2)}\frac{(p^2;p^2)_\infty}{(p;p)_\infty^2}\frac{\Theta_{p^2}(pz^2)\Theta_{p^2}(q^2)}{\Theta_{p^2}(q^2z^2)} \tag{4.6}$$

$$\frac{1}{\kappa(z^2)} = \frac{(q^4z^{-2};p,q^4)_\infty(q^2z^2;p,q^4)_\infty(pz^{-2};p,q^4)_\infty(pq^2z^2;p,q^4)_\infty}{(q^4z^2;p,q^4)_\infty(q^2z^{-2};p,q^4)_\infty(pz^2;p,q^4)_\infty(pq^2z^{-2};p,q^4)_\infty} \tag{4.7}$$

$$\tau(q^{1/2}z^{-1}) = q^{-1/2}z\frac{\Theta_{q^4}(q^2z^2)}{\Theta_{q^4}(z^2)}. \tag{4.8}$$

R satisfies the so-called quasi-periodicity property

$$R_{12}(-zp^{1/2}) = (\sigma_1\otimes 1\!\!1)^{-1}\, R_{21}(z^{-1})^{-1}\,(\sigma_1\otimes 1\!\!1). \tag{4.9}$$

It also obeys the crossing-symmetry property (A.8), but not unitarity (A.7).

This matrix defines the elliptic algebra $\mathcal{A}_{q,p}(\widehat{sl(2)_c})$ as

$$R_{12}(z_1/z_2,q,p)\,L_1(z_1)\,L_2(z_2) = L_2(z_2)\,L_1(z_1)\,R_{12}(z_1/z_2,q,p^* = pq^{-2c}). \tag{4.10}$$

4.2. Non-elliptic limit: quantum affine algebra $\mathcal{U}_q(\widehat{sl(2)_c})$

Starting from the above R-matrix of $\mathcal{A}_{q,p}(\widehat{sl(2)_c})$, and taking the limit $p \to 0$, one obtains the $\mathcal{U}_q(\widehat{sl(2)_c})$ algebra, with its R-matrix given by

$$R_V(z) = \rho(z^2) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \dfrac{q(1-z^2)}{1-q^2z^2} & \dfrac{z(1-q^2)}{1-q^2z^2} & 0 \\ 0 & \dfrac{z(1-q^2)}{1-q^2z^2} & \dfrac{q(1-z^2)}{1-q^2z^2} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{4.11}$$

The normalization factor is

$$\rho(z^2) = q^{-1/2} \frac{(q^2z^2; q^4)_\infty^2}{(z^2; q^4)_\infty (q^4z^2; q^4)_\infty}. \tag{4.12}$$

It is known [11] that the algebra $\mathcal{U}_q(\widehat{sl(2)_c})$ is only obtained after a suitable renormalization of the generators of $\mathcal{A}_{q,p}(\widehat{sl(2)_c})$ and a subsequent non-continuous limit $p \to 0$.

The algebra $\mathcal{U}_q(\widehat{sl(2)_c})$ is then defined by the relations

$$R_{12}(z_1/z_2)\, L_1^{\pm}(z_1)\, L_2^{\pm}(z_2) = L_2^{\pm}(z_2)\, L_1^{\pm}(z_1)\, R_{12}(z_1/z_2) \tag{4.13}$$

$$R_{12}(q^{c/2}z_1/z_2)\, L^+{}_1(z_1)\, L^-{}_2(z_2) = L^-{}_2(z_2)\, L^+{}_1(z_1)\, R_{12}(q^{-c/2}z_1/z_2). \tag{4.14}$$

As indicated in the introduction, we do not discuss the problem of generator expansions here. The same caveat will hold throughout the whole paper, namely we shall assume that suitable, consistent expansions of the Lax equations will exist to generate well defined algebraic structures.

4.3. Scaling limit

The so-called scaling limit of an algebra will be understood as the algebra defined by the scaling limit of the R-matrix of the initial structure. It is obtained by setting in the R-matrix $p = q^{2r}$ (elliptic nome) and $z = q^{\mathrm{i}\beta/\pi}$ (spectral parameter) with $q \to 1$ and r, β being kept fixed. The spectral parameter in the Lax operator is now to be taken as β.

4.3.1. Deformed double Yangian $\mathcal{D}Y_r^{V8}(sl(2))_c$. Taking the scaling limit of $\mathcal{A}_{q,p}(\widehat{sl(2)_c})$, one obtains the $\mathcal{D}Y_r^{V8}(sl(2))_c$ algebra. Its R-matrix takes the form [4, 41] (the superscript $V8$ is a token of the eight non-vanishing entries of the vertex-type R-matrix):

$$R_{V8}(\beta, r) = \rho_{V8}(\beta; r) \begin{pmatrix} \dfrac{\cos\frac{\mathrm{i}\beta}{2r}\cos\frac{\pi}{2r}}{\cos\frac{\pi+\mathrm{i}\beta}{2r}} & 0 & 0 & -\dfrac{\sin\frac{\mathrm{i}\beta}{2r}\sin\frac{\pi}{2r}}{\cos\frac{\pi+\mathrm{i}\beta}{2r}} \\ 0 & \dfrac{\sin\frac{\mathrm{i}\beta}{2r}\cos\frac{\pi}{2r}}{\sin\frac{\pi+\mathrm{i}\beta}{2r}} & \dfrac{\cos\frac{\mathrm{i}\beta}{2r}\sin\frac{\pi}{2r}}{\sin\frac{\pi+\mathrm{i}\beta}{2r}} & 0 \\ 0 & \dfrac{\cos\frac{\mathrm{i}\beta}{2r}\sin\frac{\pi}{2r}}{\sin\frac{\pi+\mathrm{i}\beta}{2r}} & \dfrac{\sin\frac{\mathrm{i}\beta}{2r}\cos\frac{\pi}{2r}}{\sin\frac{\pi+\mathrm{i}\beta}{2r}} & 0 \\ -\dfrac{\sin\frac{\mathrm{i}\beta}{2r}\sin\frac{\pi}{2r}}{\cos\frac{\pi+\mathrm{i}\beta}{2r}} & 0 & 0 & \dfrac{\cos\frac{\mathrm{i}\beta}{2r}\cos\frac{\pi}{2r}}{\cos\frac{\pi+\mathrm{i}\beta}{2r}} \end{pmatrix}. \tag{4.15}$$

The normalization factor is

$$\rho_{V8}(\beta;r) = -\frac{S_2(-(\mathrm{i}\beta/\pi)|r,2)\,S_2(1+(\mathrm{i}\beta/\pi)|r,2)}{S_2((\mathrm{i}\beta/\pi)|r,2)\,S_2(1-(\mathrm{i}\beta/\pi)|r,2)}\,\mathrm{cotan}\,\frac{\mathrm{i}\beta}{2}. \tag{4.16}$$

$S_2(x|\omega_1,\omega_2)$ is the Barnes double sine function (see appendix B).

This R-matrix satisfies the quasi-periodicity property

$$R_{12}(\beta - \mathrm{i}\pi r) = (\sigma_1 \otimes \mathbb{1})^{-1}\,R_{21}(-\beta)^{-1}\,(\sigma_1 \otimes \mathbb{1}) \tag{4.17}$$

where σ_1 is the usual Pauli matrix.

It also obeys the crossing-symmetry property (A.8), but not (A.7).

The algebra $\mathcal{D}Y_r^{V8}(sl(2))_c$ is then defined by the relation

$$R_{12}(\beta_1-\beta_2,r)\,L_1(\beta_1)\,L_2(\beta_2) = L_2(\beta_2)\,L_1(\beta_1)\,R_{12}(\beta_1-\beta_2,r-c). \tag{4.18}$$

4.3.2. Double Yangian $\mathcal{D}Y(sl(2))_c$. Starting now from the quantum affine algebra $\mathcal{U}_q(\widehat{sl(2)_c})$ and taking its scaling limit, one obtains the double Yangian algebra $\mathcal{D}Y(sl(2))_c$ [29]. Its R-matrix is given by

$$R(\beta) = \rho(\beta)\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \dfrac{\mathrm{i}\beta}{\mathrm{i}\beta+\pi} & \dfrac{\pi}{\mathrm{i}\beta+\pi} & 0 \\ 0 & \dfrac{\pi}{\mathrm{i}\beta+\pi} & \dfrac{\mathrm{i}\beta}{\mathrm{i}\beta+\pi} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{4.19}$$

The normalization factor is

$$\rho(\beta) = \frac{\Gamma_1((\mathrm{i}\beta/\pi)|2)\Gamma_1(2+(\mathrm{i}\beta/\pi)|2)}{\Gamma_1(1+(\mathrm{i}\beta/\pi)|2)^2}. \tag{4.20}$$

Taking the limit $r \to \infty$ of the R-matrix of $\mathcal{D}Y_r^{V8}(sl(2))_c$ (corresponding to the previous $p \to 0$ limit), one also obtains the double Yangian algebra.

Note that in both previous cases, the limit procedure may be applied directly to the Lax matrices, leading to the explicit, continuous labelled algebras, respectively denoted $\mathcal{A}_{\hbar,\eta}(\widehat{sl(2)_c})$ and $\mathcal{A}_{\hbar,0}(\widehat{sl(2)_c})$ [28].

The different limit procedures in the vertex case are summarized in figure 2.

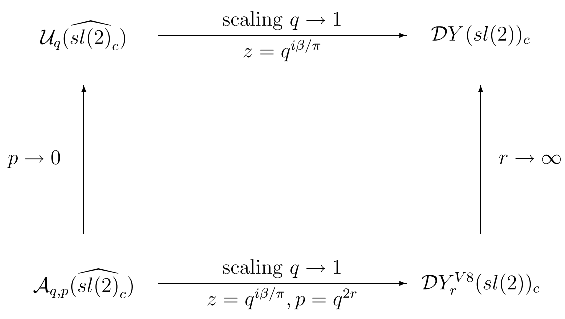

Figure 2. The vertex-case diagram: limit procedures.

4.4. Finite algebras

Up to now, the various limits led to affine structures. We now consider another kind of limit where the algebra is 'factorized'. The resulting structure is based on a finite $sl(2)$ algebra. This is interpreted as a highly degenerate consistent representation of the affine algebras at $c = 0$, where all generators are expressed in terms of only four generators.

4.4.1. Sklyanin algebra. The Sklyanin algebra [10] is constructed from $\mathcal{A}_{q,p}(\widehat{sl(2)_c})$ taken at $c = 0$. The R-matrix (4.1) can be written as

$$R(z) = \mathbb{1} \otimes \mathbb{1} + \sum_{\alpha=1}^{3} W_\alpha(z)\, \sigma_\alpha \otimes \sigma_\alpha \tag{4.21}$$

where σ_α are the Pauli matrices and $W_\alpha(z)$ are expressed in terms of the Jacobi elliptic functions. A particular z dependence of the $L(z)$ operators is chosen, leading to a factorization of the z dependence in the *RLL* relations. Indeed, setting

$$L(z) = S_0 \mathbb{1} + \sum_{\alpha=1}^{3} W_\alpha(z)\, S_\alpha \sigma_\alpha \tag{4.22}$$

one obtains an algebra with four generators S^α ($\alpha = 0, \ldots, 3$) and commutation relations

$$\begin{aligned} [S_0, S_\alpha] &= -\mathrm{i} J_{\beta\gamma}(S_\beta S_\gamma + S_\gamma S_\beta) \\ [S_\alpha, S_\beta] &= \mathrm{i}(S_0 S_\gamma + S_\gamma S_0) \end{aligned} \tag{4.23}$$

where $J_{\alpha\beta} = \frac{W_\alpha^2 - W_\beta^2}{W_\gamma^2 - 1}$ and α, β, γ are cyclic permutations of 1, 2, 3. The structure functions $J_{\alpha\beta}$ are actually independent of z. Hence we obtain an algebra where the z dependence has been dropped out.

4.4.2. $\mathcal{U}_r(sl(2))$. The same factorization procedure (4.21) and (4.22) applied to $\mathcal{D}Y_r^{V8}(sl(2))_c$ leads to a $\mathcal{U}_r(sl(2))$ algebra described by (4.23) with now $J_{12} = -J_{31} = \tan^2(\pi/2r)$ and $J_{23} = 0$. We recognize the algebra $\mathcal{U}_{q'}(sl(2))$ if we set $q' = \mathrm{e}^{\mathrm{i}\pi/r}$.

Remark. The scaling limit of the Sklyanin algebra (4.23) also leads to the algebra $\mathcal{U}_r(sl(2))$.

4.4.3. Other factorizations. Applying the factorization procedure (4.21) and (4.22) to the quantum affine algebra $\mathcal{U}_q(\widehat{sl(2)_c})$, one simply obtains the finite $\mathcal{U}_q(sl(2))$ algebra.

Let us remark that this algebra is also the $p \to 0$ limit of the Sklyanin algebra.

If we finally apply the factorization procedure to the double Yangian $\mathcal{D}Y(sl(2))_c$, one obtains $J_{\alpha\beta} = 0$. Setting the central generator S_0 to 1, we recognize the classical $\mathcal{U}(sl(2))$ algebra.

Note that $\mathcal{U}(sl(2))$ can also be viewed as:

(a) the $r \to \infty$ limit of $\mathcal{U}_r(sl(2))$;
(b) the $q \to 1$ limit ('scaling limit') of $\mathcal{U}_q(sl(2))$.

The different limit procedures in the finite vertex case are summarized in figure 3.

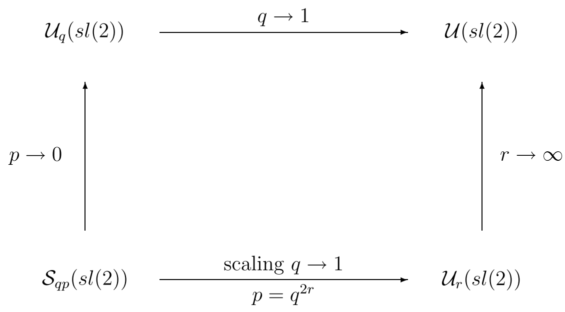

Figure 3. The finite vertex-case diagram: limit procedures.

5. Face-type algebras

5.1. Elliptic algebra $\mathcal{B}_{q,p,\lambda}(\widehat{sl(2)_c})$

The starting point in the face case is the $\mathcal{B}_{q,p,\lambda}(\widehat{sl(2)_c})$ algebra. Let $\{h, c, d\}$ be a basis of the Cartan subalgebra of $(\widehat{sl(2)_c})$. If r, s, s' are complex numbers, we set $\lambda = \frac{1}{2}(s+1)h + s'c + (r+2)d$. The elliptic parameter p and the dynamical parameter w are related to the deformation parameter q by $p = q^{2r}$, $w = q^{2s}$.

The R-matrix of $\mathcal{B}_{q,p,\lambda}(\widehat{sl(2)_c})$ is [12, 18]

$$R(z; p, w) = \rho(z; p) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & b(z) & c(z) & 0 \\ 0 & \bar{c}(z) & \bar{b}(z) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{5.1}$$

where

$$b(z) = q \frac{(pw^{-1}q^2; p)_\infty (pw^{-1}q^{-2}; p)_\infty}{(pw^{-1}; p)_\infty^2} \frac{\Theta_p(z)}{\Theta_p(q^2 z)} \tag{5.2}$$

$$\bar{b}(z) = q \frac{(wq^2; p)_\infty (wq^{-2}; p)_\infty}{(w; p)_\infty^2} \frac{\Theta_p(z)}{\Theta_p(q^2 z)} \tag{5.3}$$

$$c(z) = \frac{\Theta_p(q^2)}{\Theta_p(w)} \frac{\Theta_p(wz)}{\Theta_p(q^2 z)} \tag{5.4}$$

$$\bar{c}(z) = z \frac{\Theta_p(q^2)}{\Theta_p(pw^{-1})} \frac{\Theta_p(pw^{-1}z)}{\Theta_p(q^2 z)}. \tag{5.5}$$

The normalization factor is

$$\rho(z; p) = q^{-1/2} \frac{(q^2 z; p, q^4)_\infty^2}{(z; p, q^4)_\infty (q^4 z; p, q^4)_\infty} \frac{(pz^{-1}; p, q^4)_\infty (pq^4 z^{-1}; p, q^4)_\infty}{(pq^2 z^{-1}; p, q^4)_\infty^2}. \tag{5.6}$$

The elliptic algebra $\mathcal{B}_{q,p,\lambda}(\widehat{sl(2)_c})$ is then defined by [12, 18]

$$R_{12}(z_1/z_2, \lambda + h)\, L_1(z_1, \lambda)\, L_2(z_2, \lambda + h^{(1)}) = L_2(z_2, \lambda)\, L_1(z_1, \lambda + h^{(2)})\, R_{12}(z_1/z_2, \lambda). \tag{5.7}$$

5.2. Dynamical quantum affine algebras $\mathcal{U}_{q,\lambda}(\widehat{sl(2)_c})$

Starting from the $\mathcal{B}_{q,p,\lambda}(\widehat{sl(2)_c})$ R-matrix, and taking the limit $p \to 0$, one obtains the $\mathcal{U}_{q,\lambda}(\widehat{sl(2)_c})$ one.

The R-matrix of $\mathcal{U}_{q,\lambda}(\widehat{sl(2)_c})$ is

$$R(z;w) = \rho(z) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \dfrac{q(1-z)}{1-q^2z} & \dfrac{(1-q^2)(1-wz)}{(1-q^2z)(1-w)} & 0 \\ 0 & \dfrac{(1-q^2)(z-w)}{(1-q^2z)(1-w)} & \dfrac{q(1-z)}{(1-q^2z)} \dfrac{(1-wq^2)(1-wq^{-2})}{(1-w)^2} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{5.8}$$

The normalization factor is

$$\rho(z) = q^{-1/2} \frac{(q^2z;q^4)_\infty^2}{(z;q^4)_\infty (q^4z;q^4)_\infty}. \tag{5.9}$$

5.3. Non-dynamical limit

Taking the limit $w \to 0$ in $\mathcal{U}_{q,\lambda}(\widehat{sl(2)_c})$, one obtains the algebra $\mathcal{U}_q(\widehat{sl(2)_c})$ with an R-matrix:

$$R_F(z) = \rho(z) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \dfrac{q(1-z)}{1-q^2z} & \dfrac{1-q^2}{1-q^2z} & 0 \\ 0 & \dfrac{z(1-q^2)}{1-q^2z} & \dfrac{q(1-z)}{1-q^2z} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{5.10}$$

The normalization factor is

$$\rho(z) = q^{-1/2} \frac{(q^2z;q^4)_\infty^2}{(z;q^4)_\infty (q^4z;q^4)_\infty}. \tag{5.11}$$

Remark 1. The matrix (4.11) differs from the matrix (5.10) by rescaling $z \to z^2$ and symmetrization between the $e_{12} \otimes e_{21}$ and $e_{21} \otimes e_{12}$ terms.

Actually, the matrix $R(z)$ is computed from the universal $\mathcal{R}$-matrix of $\mathcal{U}_q(\widehat{sl(2)_c})$ by $R(z_1/z_2) = (\pi_{z_1} \otimes \pi_{z_2})\mathcal{R}$ where π is a spin-$\frac{1}{2}$ evaluation representation [39]. The evaluation parameter z is implemented using the homogeneous gradation in the representation π_z^F (respectively, the principal gradation in the representation π_z^V). In the Chevalley basis, they are given by

$$\begin{array}{llll} \pi_z^F(e_1) = e_{12} & \pi_z^F(f_1) = e_{21} & \pi_z^F(e_0) = ze_{21} & \pi_z^F(f_0) = z^{-1}e_{12} \\ \pi_z^V(e_1) = ze_{12} & \pi_z^V(f_1) = z^{-1}e_{21} & \pi_z^V(e_0) = ze_{21} & \pi_z^V(f_0) = z^{-1}e_{12}. \end{array} \tag{5.12}$$

The corresponding presentations will be denoted, respectively, $\mathcal{U}_q^F(\widehat{sl(2)_c})$ for (5.10) and $\mathcal{U}_q^V(\widehat{sl(2)_c})$ for (4.11). This change of presentation is implemented, at the level of the Lax operators, by the following change of basis:

$$L_F^-(z^2) = \begin{pmatrix} z^{-1/2} & 0 \\ 0 & z^{1/2} \end{pmatrix} L_V^-(z) \begin{pmatrix} z^{1/2} & 0 \\ 0 & z^{-1/2} \end{pmatrix} \tag{5.13}$$

$$L_F^+(z^2) = \begin{pmatrix} z^{-1/2}q^{-c/4} & 0 \\ 0 & z^{1/2}q^{c/4} \end{pmatrix} L_V^+(z) \begin{pmatrix} z^{1/2}q^{-c/4} & 0 \\ 0 & z^{-1/2}q^{c/4} \end{pmatrix}. \tag{5.14}$$

Remark 2. As in the case of (4.11), the scaling limit of the R-matrix (5.10) of $\mathcal{U}_q^F(\widehat{sl(2)_c})$ gives the R-matrix (4.19) of $\mathcal{D}Y(sl(2))_c$.

5.4. *Dynamical deformed double Yangian* $\mathcal{D}Y_{r,s}(sl(2))_c$

Starting again from the $\mathcal{B}_{q,p,\lambda}(\widehat{sl(2)_c})$ case, and now taking the scaling limit $p = q^{2r}$ (elliptic nome), $z = q^{2\mathrm{i}\beta/\pi}$ (spectral parameter), $w = q^{2s}$ (dynamical parameter) with $q \to 1$, one obtains a new structure, interpreted as a dynamical deformed centrally extended double Yangian $\mathcal{D}Y_{r,s}(sl(2))_c$.

The R-matrix of $\mathcal{D}Y_{r,s}(sl(2))_c$ is

$$R(\beta; r, s) = \rho(\beta; r) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & b(\beta) & c(\beta) & 0 \\ 0 & \bar{c}(\beta) & \bar{b}(\beta) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{5.15}$$

where

$$b(\beta) = \frac{\Gamma_1(r-s|r)^2}{\Gamma_1(r-s+1|r)\,\Gamma_1(r-s-1|r)} \frac{\sin(\mathrm{i}\beta/r)}{\sin((\pi+\mathrm{i}\beta)/r)} \tag{5.16}$$

$$c(\beta) = \frac{\sin((\pi s+\mathrm{i}\beta)/r)}{\sin(\pi s/r)} \frac{\sin(\pi/r)}{\sin((\pi+\mathrm{i}\beta)/r)} \tag{5.17}$$

$$\bar{b}(\beta) = \frac{\Gamma_1(s|r)^2}{\Gamma_1(s+1|r)\,\Gamma_1(s-1|r)} \frac{\sin(\mathrm{i}\beta/r)}{\sin((\pi+\mathrm{i}\beta)/r)} \tag{5.18}$$

$$\bar{c}(\beta) = \frac{\sin((\pi s-\mathrm{i}\beta)/r)}{\sin(\pi s/r)} \frac{\sin(\pi/r)}{\sin((\pi+\mathrm{i}\beta)/r)}. \tag{5.19}$$

The normalization factor is the same as formula (4.16), rewritten as

$$\rho(\beta; r) = \frac{S_2^2(1+(\mathrm{i}\beta/\pi)|r,2)}{S_2((\mathrm{i}\beta/\pi)|r,2)\,S_2(2+(\mathrm{i}\beta/\pi)|r,2)}. \tag{5.20}$$

This new algebra $\mathcal{D}Y_{r,s}(sl(2))_c$ is then defined by the relations

$$R_{12}(\beta_1-\beta_2, \lambda+h)\,L_1(\beta_1,\lambda)\,L_2(\beta_2,\lambda+h^{(1)}) = L_2(\beta_2,\lambda)\,L_1(\beta_1,\lambda+h^{(2)})\,R_{12}(\beta_1-\beta_2,\lambda). \tag{5.21}$$

5.5. *Dynamical double Yangian* $\mathcal{D}Y_s(sl(2))_c$

Taking the limit $r \to \infty$ in $\mathcal{D}Y_{r,s}(sl(2))_c$, one obtains a new, dynamical, centrally extended double Yangian $\mathcal{D}Y_s(sl(2))_c$.

The R-matrix of $\mathcal{D}Y_s(sl(2))_c$ is given by

$$R(\beta) = \rho(\beta) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \dfrac{\mathrm{i}\beta}{\mathrm{i}\beta+\pi} & \dfrac{\pi s+\mathrm{i}\beta}{s(\mathrm{i}\beta+\pi)} & 0 \\ 0 & \dfrac{\pi s-\mathrm{i}\beta}{s(\mathrm{i}\beta+\pi)} & \dfrac{s^2-1}{s^2}\dfrac{\mathrm{i}\beta}{\mathrm{i}\beta+\pi} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{5.22}$$

The normalization factor is

$$\rho(\beta) = \frac{\Gamma_1((\mathrm{i}\beta/\pi)|2)\Gamma_1(2+(\mathrm{i}\beta/\pi)|2)}{\Gamma_1(1+(\mathrm{i}\beta/\pi)|2)^2}. \tag{5.23}$$

Remark 1. This R-matrix (5.22) is also obtained by taking the scaling limit of the R-matrix (5.8) of $\mathcal{U}_{q,\lambda}(\widehat{sl(2)_c})$.

Remark 2. The $|s| \to \infty$ limit of the R-matrix (5.22) gives back the R-matrix (4.19) of $\mathcal{D}Y(sl(2))_c$.

5.6. *Dynamical deformed double Yangian $\mathcal{D}Y_{r,s}^{-\infty}(sl(2))_c$ in the trigonometric limit*

Starting again from $\mathcal{D}Y_{r,s}(sl(2))_c$ and taking $s \ll 0$, but retaining the oscillating behaviour in s, one obtains $\mathcal{D}Y_{r,s}^{-\infty}(sl(2))_c$, another dynamical deformed centrally extended double Yangian structure.

The R-matrix of $\mathcal{D}Y_{r,s}^{-\infty}(sl(2))_c$ reads

$$R(\beta;r,s) = \rho(\beta;r) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \dfrac{\sin\frac{\mathrm{i}\beta}{r}}{\sin\frac{\pi+\mathrm{i}\beta}{r}} & \dfrac{\sin\frac{\pi s+\mathrm{i}\beta}{r}}{\sin\frac{\pi s}{r}}\dfrac{\sin\frac{\pi}{r}}{\sin\frac{\pi+\mathrm{i}\beta}{r}} & 0 \\ 0 & \dfrac{\sin\frac{\pi s-\mathrm{i}\beta}{r}}{\sin\frac{\pi s}{r}}\dfrac{\sin\frac{\pi}{r}}{\sin\frac{\pi+\mathrm{i}\beta}{r}} & \dfrac{\sin\pi\frac{s+1}{r}\sin\pi\frac{s-1}{r}}{\sin^2\frac{\pi s}{r}}\dfrac{\sin\frac{\mathrm{i}\beta}{r}}{\sin\frac{\pi+\mathrm{i}\beta}{r}} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{5.24}$$

The normalization factor is the same as for $\mathcal{D}Y_{r,s}(sl(2))_c$, see (5.20).

Remark 1. The limit $r \to \infty$ of the R-matrix (5.24) again gives the R-matrix (5.22) of $\mathcal{D}Y_s(sl(2))_c$.

Remark 2. Correspondence with $\mathcal{U}_{q,\lambda}(\widehat{sl(2)_c})$. The previous R-matrix is proportional to that of $\mathcal{U}_{q,\lambda}(\widehat{sl(2)_c})$ through the following identifications of parameters:

$$z = \mathrm{e}^{-2\beta/r} \qquad q = \mathrm{e}^{\mathrm{i}\pi/r} \qquad w = \mathrm{e}^{2\mathrm{i}\pi s/r}. \tag{5.25}$$

The same identification of parameters applied to the R-matrix (5.15) of $\mathcal{D}Y_{r,s}(sl(2))_c$ leads to an R-matrix close to that of $\mathcal{U}_{q,\lambda}(\widehat{sl(2)_c})$, but with Γ-function dependence in the dynamical parameter. This would define a new dynamical algebraic structure $\mathcal{U}_{q,\lambda}^{\Gamma}(\widehat{sl(2)_c})$.

5.7. Deformed double Yangian $\mathcal{D}Y_r^F(sl(2))_c$

Now taking the limit $s \to \mathrm{i}\infty$ in $\mathcal{D}Y_{r,s}(sl(2))_c$, one obtains a non-dynamical structure $\mathcal{D}Y_r^F(sl(2))_c$.

The R-matrix of $\mathcal{D}Y_r^F(sl(2))_c$ is given by

$$R(\beta;r) = \rho(\beta;r)\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \dfrac{\sin(\mathrm{i}\beta/r)}{\sin((\pi+\mathrm{i}\beta)/r)} & \mathrm{e}^{\beta/r}\dfrac{\sin(\pi/r)}{\sin((\pi+\mathrm{i}\beta)/r)} & 0 \\ 0 & \mathrm{e}^{-\beta/r}\dfrac{\sin(\pi/r)}{\sin((\pi+\mathrm{i}\beta)/r)} & \dfrac{\sin(\mathrm{i}\beta/r)}{\sin((\pi+\mathrm{i}\beta)/r)} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{5.26}$$

The normalization factor is the same as for $\mathcal{D}Y_{r,s}(sl(2))_c$.

Remark 1. The limit $r \to \infty$ of the R-matrix (5.26) gives again the R-matrix of $\mathcal{D}Y(sl(2))_c$.

Remark 2. Correspondence with $\mathcal{U}_q(\widehat{sl(2)}_c)$. This matrix is proportional to that of $\mathcal{U}_q(\widehat{sl(2)}_c)$ (equation (5.10)) through the following identifications of parameters:

$$z = \mathrm{e}^{-2\beta/r} \qquad q = \mathrm{e}^{\mathrm{i}\pi/r}. \tag{5.27}$$

The different limit procedures in the face case are summarized in figure 4.

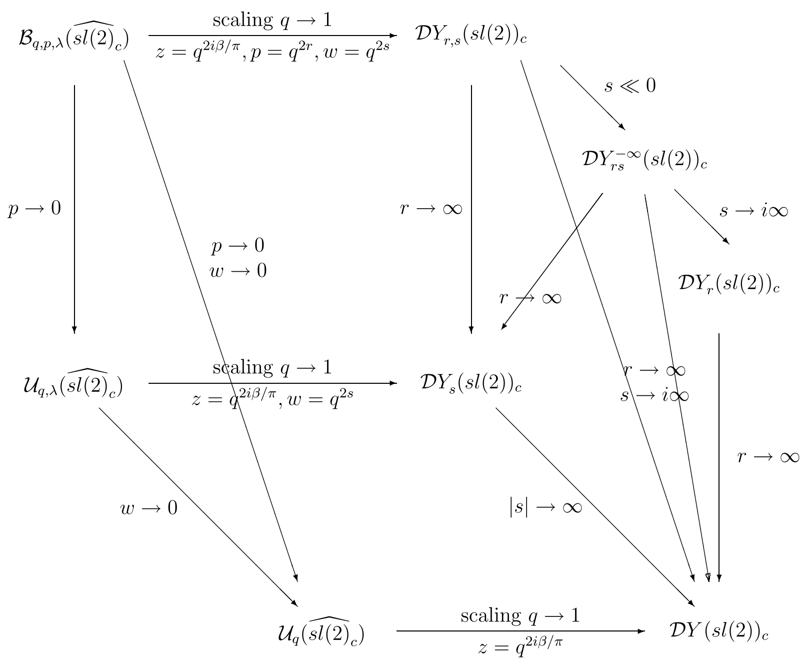

Figure 4. The face-case diagram: limit procedures.

5.8. Finite-dimensional algebras

In contrast with the vertex case, the finite face-type elliptic algebras have not yet been obtained from the affine algebras by a factorization procedure. The starting point of our description will therefore be the R-matrix representation of $\mathcal{B}_{q,\lambda}(sl(2))$ given in [40].

5.8.1. Elliptic algebra $\mathcal{B}_{q,\lambda}(sl(2))$. The R-matrix of $\mathcal{B}_{q,\lambda}(sl(2))$ is

$$R(w) = q^{-1/2}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & q & \dfrac{1-q^2}{1-w} & 0 \\ 0 & -\dfrac{w(1-q^2)}{1-w} & \dfrac{q(1-wq^2)(1-wq^{-2})}{(1-w)^2} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{5.28}$$

Remark. The limit $w \to 0$ of this matrix gives the usual R-matrix of $\mathcal{U}_q(sl(2))$

$$R = q^{-1/2}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & q & 1-q^2 & 0 \\ 0 & 0 & q & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{5.29}$$

5.8.2. Dynamical algebra $\mathcal{U}_s(sl(2))$. Taking the scaling limit $w = q^{2s}$ with $q \to 1$, we obtain the dynamical algebra $\mathcal{U}_s(sl(2))$ with the R-matrix

$$R = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & s^{-1} & 0 \\ 0 & -s^{-1} & 1-s^{-2} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{5.30}$$

The limit $|s| \to \infty$ of (5.30) gives $\mathbb{1}$, which is the evaluated R-matrix of $\mathcal{U}(sl(2))$. In section 7.8.2 we prove that the R-matrix (5.30) is the representation of a universal R-matrix $\mathcal{R}_{\mathcal{U}_s}$ obtained through a twist procedure from $\mathbb{1}$. Taking $L = (\pi \otimes \mathrm{i}d)(\mathcal{R}_{\mathcal{U}_s})$, i.e.

$$L = \begin{pmatrix} 1 & s^{-1}f \\ (1-s-h)^{-1}e & 1+s^{-1}(1-s-h)^{-1}ef \end{pmatrix} \tag{5.31}$$

we have checked that the dynamical RLL relations ((5.21) without a β dependence) are provided by the commutation relations of the $\mathcal{U}(sl(2))$ generators e, f and h.

Twist operations

We now describe the twist connections between the various algebraic structures previously defined. We first discuss twist-like actions between vertex-type algebras. We then give the TLA between face-like algebras. The TLAs are classified here according to the 'arrival' algebraic structure, i.e. with the highest number of parameters.

6. Vertex-type algebras

6.1. Twist operator $\mathcal{U}_q^V(\widehat{sl(2)_c}) \to \mathcal{A}_{q,p}(\widehat{sl(2)_c})$

The existence of a twist operator between $\mathcal{U}_q^V(\widehat{sl(2)_c})$ and $\mathcal{A}_{q,p}(\widehat{sl(2)_c})$ was proved at the level of universal matrices in [12]. Once the operators are evaluated, one obtains

$$R[\mathcal{A}_{q,p}(\widehat{sl(2)_c})] = E_{21}^{(1)}(z^{-1};\, p)R[\mathcal{U}_q^V(\widehat{sl(2)_c})]E_{12}^{(1)}(z;\, p)^{-1}. \tag{6.1}$$

The twist operator $E^{(1)}(z;\, p)$ is given by [34, 35]

$$E^{(1)}(z;\, p) = \rho_E(z;\, p)\begin{pmatrix} a_E(z) & 0 & 0 & d_E(z) \\ 0 & b_E(z) & c_E(z) & 0 \\ 0 & c_E(z) & b_E(z) & 0 \\ d_E(z) & 0 & 0 & a_E(z) \end{pmatrix} \tag{6.2}$$

where

$$a_E(z) \pm d_E(z) = \frac{(\mp p^{1/2}qz;\, p)_\infty}{(\mp p^{1/2}q^{-1}z;\, p)_\infty} \tag{6.3}$$

$$b_E(z) \pm c_E(z) = \frac{(\mp pqz;\, p)_\infty}{(\mp pq^{-1}z;\, p)_\infty}. \tag{6.4}$$

The normalization factor is

$$\rho_E(z;\, p) = \frac{(pz^2;\, p, q^4)_\infty (pq^4z^2;\, p, q^4)_\infty}{(pq^2z^2;\, p, q^4)_\infty^2}. \tag{6.5}$$

6.2. Deformed double Yangians $\mathcal{D}Y_r^V(sl(2))_c$

6.2.1. Deformed double Yangian $\mathcal{D}Y_r^{V6}(sl(2))_c$. We need to define an algebraic structure not previously derived in this paper.

The R-matrix (4.15) of the deformed double Yangian $\mathcal{D}Y_r^{V8}(sl(2))_c$ can be related to the two-body S matrix of the sine–Gordon theory $S_{SG}(\beta, r)$ by a rigid twist operator. The connection goes as follows. One defines the following R-matrix [4]:

$$\begin{aligned} R_{V6}(\beta, r) &= \mathrm{cotan}(\tfrac{1}{2}\mathrm{i}\beta)S_{SG}(\beta, r) \\ &= \rho_{V6}(\beta;\, r)\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \dfrac{\sin(\mathrm{i}\beta/r)}{\sin((\pi+\mathrm{i}\beta)/r)} & \dfrac{\sin(\pi/r)}{\sin(\pi+\mathrm{i}\beta)/r)} & 0 \\ 0 & \dfrac{\sin(\pi/r)}{\sin((\pi+\mathrm{i}\beta)/r)} & \dfrac{\sin(\mathrm{i}\beta/r)}{\sin((\pi+\mathrm{i}\beta)/r)} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{aligned} \tag{6.6}$$

where $\rho_{V6}(\beta;\, r) = \rho_{V8}(\beta;\, r)$, see (4.16). This R-matrix is assumed to define by the *RLL* formalism an algebraic structure denoted by $\mathcal{D}Y_r^{V6}(sl(2))_c$.

One now has

$$R[\mathcal{D}Y_r^{V8}(sl(2))_c] = K_{21}R[\mathcal{D}Y_r^{V6}(sl(2))_c]K_{12}^{-1}. \tag{6.7}$$

The rigid twist operator K is given by

$$K = V \otimes V \qquad \text{with} \quad V = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}. \tag{6.8}$$

This implies an isomorphism between $\mathcal{D}Y_r^{V8}(sl(2))_c$ and $\mathcal{D}Y_r^{V6}(sl(2))_c$ where the Lax operators are connected by $L_{V8} = VL_{V6}V^{-1}$.

Remark 1. The rigid twist leaves invariant the R-matrix of the undeformed double Yangian, upon which V induces an automorphism.

Remark 2. The R-matrix (6.6) is proportional to that of $\mathcal{U}_q^V(\widehat{sl(2)}_c)$ (equation (4.11)) through the following identifications of parameters:

$$z = \mathrm{e}^{-\beta/r} \qquad q = \mathrm{e}^{\mathrm{i}\pi/r}. \tag{6.9}$$

6.2.2. Twist operator $\mathcal{D}Y(sl(2))_c \to \mathcal{D}Y_r^{V8}(sl(2))_c$. The R-matrix of $\mathcal{D}Y_r^{V8}(sl(2))_c$ can be obtained from the R-matrix of $\mathcal{D}Y(sl(2))_c$ by a twist-like action:

$$R[\mathcal{D}Y_r^{V8}(sl(2))_c] = E_{21}^{(2)}(-\beta; r)R[\mathcal{D}Y(sl(2))_c]E_{12}^{(2)}(\beta; r)^{-1}. \tag{6.10}$$

The twist operator $E^{(2)}(\beta; r)$ is the scaling limit of the twist operator $E^{(1)}(z, p)$, see equation (6.2). It is given by

$$E^{(2)}(\beta; r) = \rho_E(\beta; r)\begin{pmatrix} a_E(\beta) & 0 & 0 & d_E(\beta) \\ 0 & b_E(\beta) & c_E(\beta) & 0 \\ 0 & c_E(\beta) & b_E(\beta) & 0 \\ d_E(\beta) & 0 & 0 & a_E(\beta) \end{pmatrix} \tag{6.11}$$

where

$$a_E(\beta) + d_E(\beta) = 1 \tag{6.12}$$

$$a_E(\beta) - d_E(\beta) = \frac{\Gamma_1(r - 1 + (\mathrm{i}\beta/\pi)|2r)}{\Gamma_1(r + 1 + (\mathrm{i}\beta/\pi)|2r)} \tag{6.13}$$

$$b_E(\beta) + c_E(\beta) = 1 \tag{6.14}$$

$$b_E(\beta) - c_E(\beta) = \frac{\Gamma_1(2r - 1 + (\mathrm{i}\beta/\pi)|2r)}{\Gamma_1(2r + 1 + (\mathrm{i}\beta/\pi)|2r)}. \tag{6.15}$$

The normalization factor is

$$\rho_E(\beta; r) = \frac{\Gamma_2(r + 1 + (\mathrm{i}\beta/\pi)|r, 2)^2}{\Gamma_2(r + (\mathrm{i}\beta/\pi)|r, 2)\Gamma_2(r + 2 + (\mathrm{i}\beta/\pi)|r, 2)}. \tag{6.16}$$

6.2.3. Twist operator $\mathcal{D}Y(sl(2))_c \to \mathcal{D}Y_r^{V6}(sl(2))_c$. Combining the previous two twist-like actions, one obtains

$$R[\mathcal{D}Y_r^{V6}(sl(2))_c] = E_{21}^{(3)}(-\beta; r)R[\mathcal{D}Y(sl(2))_c]E_{12}^{(3)}(\beta; r)^{-1}. \tag{6.17}$$

The twist operator $E^{(3)}(\beta; r)$ can be taken as $E^{(3)} = K^{-1}E^{(2)}$ which is a matrix with 16 non-vanishing entries. Noticing that $KR[\mathcal{D}Y(sl(2))_c]K^{-1} = R[\mathcal{D}Y(sl(2))_c]$, from now on we use the twist operator $E^{(3)} = K^{-1}E^{(2)}K$ which has the following form:

$$E^{(3)}(\beta; r) = \rho_E(\beta; r)\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & b'_E(\beta) & c'_E(\beta) & 0 \\ 0 & c'_E(\beta) & b'_E(\beta) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{6.18}$$

where

$$b'_E(\beta) + c'_E(\beta) = \frac{\Gamma_1(r-1+(\mathrm{i}\beta/\pi)|2r)}{\Gamma_1(r+1+(\mathrm{i}\beta/\pi)|2r)} \tag{6.19}$$

$$b'_E(\beta) - c'_E(\beta) = \frac{\Gamma_1(2r-1+(\mathrm{i}\beta/\pi)|2r)}{\Gamma_1(2r+1+(\mathrm{i}\beta/\pi)|2r)}. \tag{6.20}$$

The different twist procedures in the vertex case are summarized in figure 5.

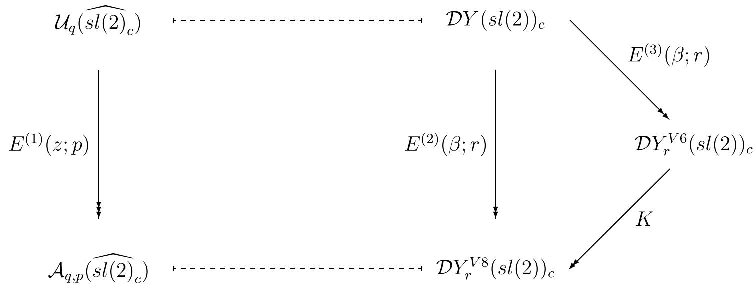

Figure 5. The vertex-case diagram: twist procedures.

7. Face-type algebras

7.1. Twist operator $\mathcal{U}_q^F(\widehat{sl(2)}_c) \to \mathcal{U}_{q,\lambda}(\widehat{sl(2)}_c)$

The two R-matrices of $\mathcal{U}_q^F(\widehat{sl(2)}_c)$ and $\mathcal{U}_{q,\lambda}(\widehat{sl(2)}_c)$ can be related by a twist operator:

$$R[\mathcal{U}_{q,\lambda}(\widehat{sl(2)}_c)] = F_{21}^{(3)}(w)R[\mathcal{U}_q^F(\widehat{sl(2)}_c)]F_{12}^{(3)}(w)^{-1}. \tag{7.1}$$

The twist operator $F^{(3)}(w)$ is given by

$$F^{(3)}(w) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & \dfrac{w(q-q^{-1})}{1-w} & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{7.2}$$

7.2. Dynamical elliptic affine algebra $\mathcal{B}_{q,p,\lambda}(\widehat{sl(2)}_c)$

7.2.1. Twist operator $\mathcal{U}_q^F(\widehat{sl(2)}_c) \to \mathcal{B}_{q,p,\lambda}(\widehat{sl(2)}_c)$. The existence of a twist operator between $\mathcal{U}_q^F(\widehat{sl(2)}_c)$ and $\mathcal{B}_{q,p,\lambda}(\widehat{sl(2)}_c)$ was proved at the level of universal matrices in [12]. Once the operators are evaluated, one obtains

$$R[\mathcal{B}_{q,p,\lambda}(\widehat{sl(2)}_c)] = F_{21}^{(1)}(z^{-1};p,w)R[\mathcal{U}_q^F(\widehat{sl(2)}_c)]F_{12}^{(1)}(z;p,w)^{-1}. \tag{7.3}$$

The twist operator $F^{(1)}(z;p,w)$ is given by

$$F^{(1)}(z;p,w) = \rho_F(z;p)\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & X_{11}(z) & X_{12}(z) & 0 \\ 0 & X_{21}(z) & X_{22}(z) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{7.4}$$

where

$$X_{11}(z) = {}_2\phi_1\left(\begin{matrix} wq^2 & q^2 \\ & w \end{matrix}; p, pq^{-2}z\right) \tag{7.5}$$

$$X_{12}(z) = \frac{w(q-q^{-1})}{1-w}{}_2\phi_1\left(\begin{matrix} wq^2 & pq^2 \\ & pw \end{matrix}; p, pq^{-2}z\right) \tag{7.6}$$

$$X_{21}(z) = z\frac{pw^{-1}(q-q^{-1})}{1-pw^{-1}}{}_2\phi_1\left(\begin{matrix} pw^{-1}q^2 & pq^2 \\ & p^2w^{-1} \end{matrix}; p, pq^{-2}z\right) \tag{7.7}$$

$$X_{22}(z) = {}_2\phi_1\left(\begin{matrix} pw^{-1}q^2 & q^2 \\ & pw^{-1} \end{matrix}; p, pq^{-2}z\right). \tag{7.8}$$

The normalization factor is

$$\rho_F(z;p) = \frac{(pz;p,q^4)_\infty(pq^4z;p,q^4)_\infty}{(pq^2z;p,q^4)^2_\infty}. \tag{7.9}$$

7.2.2. Twist operator $\mathcal{U}_{q,\lambda}(\widehat{sl(2)_c}) \to \mathcal{B}_{q,p,\lambda}(\widehat{sl(2)_c})$. Combining the last two twist-like actions, one obtains

$$R[\mathcal{B}_{q,p,\lambda}(\widehat{sl(2)_c})] = F^{(5)}_{21}(z^{-1};p,w)R[\mathcal{U}_{q,\lambda}(\widehat{sl(2)_c})]F^{(5)}_{12}(z;p,w)^{-1}. \tag{7.10}$$

The twist operator $F^{(5)}(z;p,w)$ is given by $F^{(5)} = F^{(1)}F^{(3)^{-1}}$. The normalization factor $\rho_F(z;p)$ is given by (7.9).

7.3. Dynamical double Yangian $\mathcal{D}Y_s(sl(2))_c$

By taking the scaling limit of the connection (7.1), one obtains

$$R[\mathcal{D}Y_s(sl(2))_c] = F^{(4)}_{21}(s)R[\mathcal{D}Y(sl(2))_c]F^{(4)}_{12}(s)^{-1}. \tag{7.11}$$

The twist operator $F^{(4)}(s)$ is the scaling limit of the twist operator $F^{(3)}$, see equation (7.2). It is given by

$$F^{(4)}(s) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -s^{-1} & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{7.12}$$

7.4. Deformed double Yangian $\mathcal{D}Y^F_r(sl(2))_c$

7.4.1. Twist operator $\mathcal{D}Y^V_r(sl(2))_c \to \mathcal{D}Y^F_r(sl(2))_c$. The two deformed double Yangians $\mathcal{D}Y^V_r(sl(2))_c$ and $\mathcal{D}Y^F_r(sl(2))_c$ obtained from the vertex-type algebras, on one hand, and from face-type algebras, on the other hand, are related by twist-like actions. One has

$$R[\mathcal{D}Y^F_r(sl(2))_c](\beta) = K^{(6)}_{21}(-\beta)R[\mathcal{D}Y^{V6}_r(sl(2))_c](\beta)K^{(6)}_{12}(\beta)^{-1}. \tag{7.13}$$

The twist operator $K^{(6)}$ is

$$K^{(6)}(\beta) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & e^{\beta/2r} & 0 & 0 \\ 0 & 0 & e^{-\beta/2r} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{7.14}$$

Using the rigid twist operator (6.8), one also obtains

$$R[\mathcal{D}Y_r^F(sl(2))_c](\beta) = K_{21}^{(8)}(-\beta)R[\mathcal{D}Y_r^{V8}(sl(2))_c](\beta)K_{12}^{(8)}(\beta)^{-1}. \tag{7.15}$$

The twist operator $K^{(8)}$ is given by $K^{(8)} = K^{(6)}K^{-1}$. This twist provides a link between the face-type and vertex-type double Yangian structures.

7.4.2. Twist operator $\mathcal{D}Y(sl(2))_c \to \mathcal{D}Y_r^F(sl(2))_c$. The connection between R-matrices of $\mathcal{D}Y(sl(2))_c$ and $\mathcal{D}Y_r^F(sl(2))_c$ can be established by three different combinations of previously constructed twist-like actions. These three combinations of course, by construction, give the same twist operator $F^{(7)} = K^{(6)}E^{(3)} = K^{(6)}K^{-1}E^{(2)}K$. One therefore has

$$R[\mathcal{D}Y_r^F(sl(2))_c] = F_{21}^{(7)}(-\beta; r,)R[\mathcal{D}Y(sl(2))_c]F_{12}^{(7)}(\beta; r)^{-1}. \tag{7.16}$$

7.5. Trigonometric dynamical deformed double Yangian $\mathcal{D}Y_{r,s}^{-\infty}(sl(2))_c$

7.5.1. Twist operator $\mathcal{D}Y_r^F(sl(2))_c \to \mathcal{D}Y_{r,s}^{-\infty}(sl(2))_c$. The connection between $\mathcal{D}Y_r^F(sl(2))_c$ and $\mathcal{D}Y_{r,s}^{-\infty}(sl(2))_c$ is achieved by the twist operator $F^{(3)}$:

$$R[\mathcal{D}Y_{r,s}^{-\infty}(sl(2))_c] = F_{21}^{(3)}(s)R[\mathcal{D}Y_r^F(sl(2))_c]F_{12}^{(3)}(s)^{-1}. \tag{7.17}$$

The twist operator $F^{(3)}$ is actually equal to the twist operator (7.2) by setting $q = e^{i\pi/r}$ and $w = e^{2i\pi s/r}$:

$$F^{(3)}(s) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -e^{i\pi s/r}\dfrac{\sin(\pi/r)}{\sin(\pi s/r)} & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{7.18}$$

7.5.2. Twist operator $\mathcal{D}Y(sl(2))_c \to \mathcal{D}Y_{r,s}^{-\infty}(sl(2))_c$. The combination of two twist-like operations yields the connection between $\mathcal{D}Y(sl(2))_c$ and $\mathcal{D}Y_{r,s}^{-\infty}(sl(2))_c$:

$$R[\mathcal{D}Y_{r,s}^{-\infty}(sl(2))_c] = F_{21}^{(10)}(-\beta; r, s)R[\mathcal{D}Y(sl(2))_c]F_{12}^{(10)}(\beta; r, s)^{-1}. \tag{7.19}$$

The twist operator $F^{(10)}$ is given by $F^{(10)} = F^{(3)}F^{(7)}$.

7.5.3. Twist operator $\mathcal{D}Y_s(sl(2))_c \to \mathcal{D}Y_{r,s}^{-\infty}(sl(2))_c$. Again, the combination of two twist-like operations yields the connection between $\mathcal{D}Y_s(sl(2))_c$ and $\mathcal{D}Y_{r,s}^{-\infty}(sl(2))_c$:

$$R[\mathcal{D}Y_{r,s}^{-\infty}(sl(2))_c] = F_{21}^{(8)}(-\beta; r, s)R[\mathcal{D}Y_s(sl(2))_c]F_{12}^{(8)}(\beta; r, s)^{-1}. \tag{7.20}$$

The twist operator $F^{(8)}$ is given by $F^{(8)} = F^{(10)}F^{(4)^{-1}}$.

7.6. Dynamical deformed double Yangian $\mathcal{D}Y_{r,s}(sl(2))_c$

7.6.1. Twist operator $\mathcal{D}Y_{r,s}^{-\infty}(sl(2))_c \to \mathcal{D}Y_{r,s}(sl(2))_c$. The R-matrices of $\mathcal{D}Y_{r,s}^{-\infty}(sl(2))_c$ and $\mathcal{D}Y_{r,s}(sl(2))_c$ are connected by a diagonal TLA (not depending on the spectral parameter):

$$R[\mathcal{D}Y_{r,s}(sl(2))_c] = G_{21}(r,s)R[\mathcal{D}Y_{r,s}^{-\infty}(sl(2))_c]G_{12}(r,s)^{-1}. \tag{7.21}$$

The twist operator G is given by

$$G(r,s) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & g^{-1} & 0 & 0 \\ 0 & 0 & g & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & g \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & g^{-1} \end{pmatrix} \tag{7.22}$$

where

$$g(r,s) = \frac{\Gamma_1(r-s|r)}{\Gamma_1(r-s+1|r)^{1/2}\Gamma_1(r-s-1|r)^{1/2}}. \tag{7.23}$$

Remark. Equivalently, G expressed in terms of the parameters $q = \mathrm{e}^{\mathrm{i}\pi/r}$ and $w = \mathrm{e}^{2\mathrm{i}\pi s/r}$ realizes a TLA between $\mathcal{U}_{q,\lambda}(\widehat{sl(2)}_c)$ and $\mathcal{U}_{q,\lambda}^{\Gamma}(\widehat{sl(2)}_c)$ defined in remark 2, section 5.6.

7.6.2. Twist operator $\mathcal{D}Y_s(sl(2))_c \to \mathcal{D}Y_{r,s}(sl(2))_c$. Combining the last two twists, one obtains

$$R[\mathcal{D}Y_{r,s}(sl(2))_c] = F_{21}^{(6)}(-\beta;r,s)R[\mathcal{D}Y_s(sl(2))_c]F_{12}^{(6)}(\beta;r,s)^{-1}. \tag{7.24}$$

The twist operator $F^{(6)}$ is given by $F^{(6)} = GF^{(8)}$.

7.6.3. Twist operator $\mathcal{D}Y(sl(2))_c \to \mathcal{D}Y_{r,s}(sl(2))_c$. Similarly, by a combination of previous twists, one obtains

$$R[\mathcal{D}Y_{r,s}(sl(2))_c] = F_{21}^{(2)}(-\beta;r,s)R[\mathcal{D}Y(sl(2))_c]F_{12}^{(2)}(\beta;r,s)^{-1}. \tag{7.25}$$

The twist operator $F^{(2)}$ is given by $F^{(2)} = GF^{(10)}$.

7.6.4. Twist operator $\mathcal{D}Y_r^F(sl(2))_c \to \mathcal{D}Y_{r,s}(sl(2))_c$. Finally, the connection between $\mathcal{D}Y_r^F(sl(2))_c$ and $\mathcal{D}Y_{r,s}(sl(2))_c$ is provided by

$$R[\mathcal{D}Y_{r,s}(sl(2))_c] = F_{21}^{(11)}(r,s)R[\mathcal{D}Y_r^F(sl(2))_c]F_{12}^{(11)}(r,s)^{-1}. \tag{7.26}$$

The twist operator $F^{(11)}$ is given by $F^{(11)} = GF^{(3)}$, that is

$$F^{(11)}(r,s) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & g^{-1} & -\mathrm{e}^{\mathrm{i}\pi s/r}\dfrac{\sin(\pi/r)}{\sin(\pi s/r)}g^{-1} & 0 \\ 0 & 0 & g & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{7.27}$$

where g is given by (7.23).

The different twist procedures in the face case are summarized in figure 6.

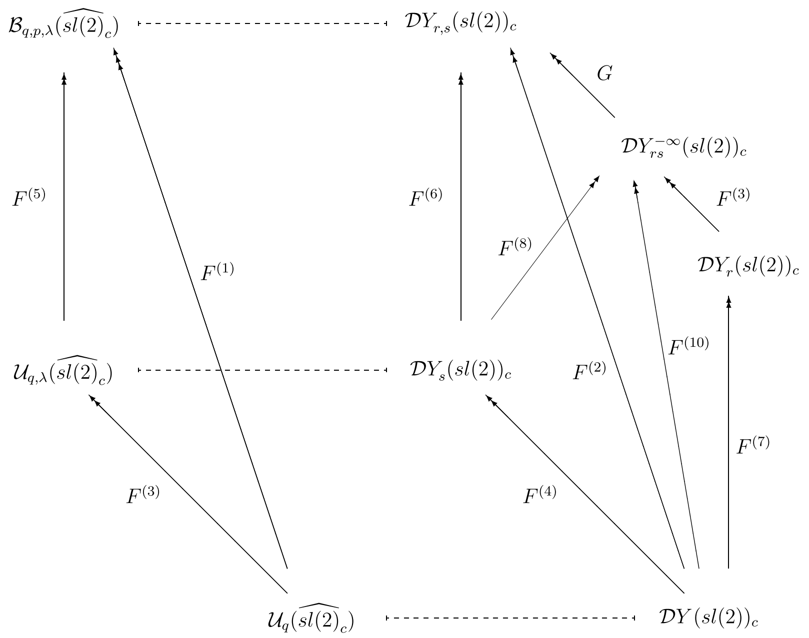

Figure 6. The face-case diagram: twist procedures.

7.7. Connections with $\mathcal{A}_{q,p;\pi}(\widehat{sl(2)_c})$ and derived algebras

7.7.1. Twist $\mathcal{B}_{q,p,\lambda}(\widehat{sl(2)_c}) \to \mathcal{A}_{q,p;\pi}(\widehat{sl(2)_c})$. The R-matrix of $\mathcal{A}_{q,p;\pi}(\widehat{sl(2)_c})$ given in [24] (actually their R^+-matrix) is

$$R = z^{1/2r}\rho(z)\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \dfrac{\Theta_p(z)\Theta_p(q^{-2}w)}{\Theta_p(q^2z)\Theta_p(w)} & \dfrac{\Theta_p(zw)\Theta_p(q^2)}{\Theta_p(q^2z)\Theta_p(w)} & 0 \\ 0 & \dfrac{\Theta_p(z^{-1}w)\Theta_p(q^2)}{\Theta_p(q^2z)\Theta_p(w)} & \dfrac{\Theta_p(z)\Theta_p(q^2w)}{\Theta_p(q^2z)\Theta_p(w)} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{7.28}$$

where $\rho(z)$ is the same as (5.6). This R-matrix is obtained from (5.1) by exchanging factors in b and $\bar{b}$ so as to reconstruct a full Θ-function dependence and correcting the $z^{1/2r}$ factor. All this can be achieved by a factorized diagonal twist which has the same form as G (7.22).

7.7.2. Twist $\mathcal{D}Y_{r,s}^{-\infty}(sl(2))_c \to \mathcal{A}_{\hbar,\eta;\pi}(\widehat{sl(2)_c})$. Again, the R-matrix of the scaling limit $\mathcal{A}_{\hbar,\eta;\pi}(\widehat{sl(2)_c})$ [24] of $\mathcal{A}_{q,p;\pi}(\widehat{sl(2)_c})$ can be obtained from that of $\mathcal{D}Y_{r,s}^{-\infty}(sl(2))_c$ (5.24) by a factorized diagonal twist. It also has the form of G (7.22), with now

$$g^2 = \frac{\sin\pi(s-1)/r}{\sin\pi s/r}.$$

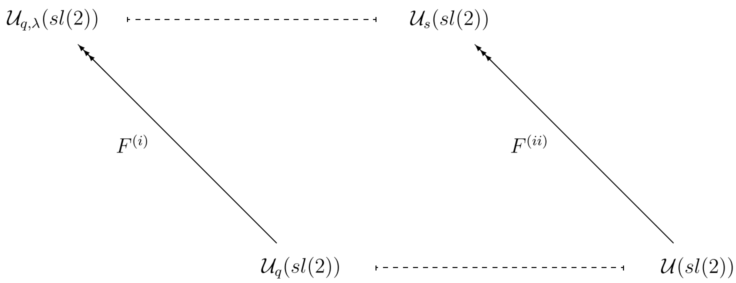

Figure 7. The finite face-case diagram: twist procedures.

7.8. *Finite-dimensional algebras*

In both cases where TLA actions are known for non-affine algebras, they are evaluations of universal twists.

7.8.1. Elliptic algebra $\mathcal{B}_{q,\lambda}(sl(2))$. The twist operator that links $\mathcal{U}_q(sl(2))$ to $\mathcal{B}_{q,\lambda}(sl(2))$ is [40]:

$$F^{(i)}(w) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & \dfrac{(q-q^{-1})w}{1-w} & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{7.29}$$

The universal form of the twist is [40]

$$\mathcal{F}(w) = \sum_{n=0}^{\infty} \frac{(q^2 w)^n (q-q^{-1})^n}{(n)_{q^{-2}}!(q^{-2}w(t^2 \otimes 1); q^{-2})_n} (et)^n \otimes (tf)^n \tag{7.30}$$

where e, f, $t^{\pm 1}$ are the generators of $\mathcal{U}_q(sl(2))$.

7.8.2. Dynamical algebra $\mathcal{U}_s(sl(2))$. Its R-matrix can be obtained by action of the twist

$$F^{(ii)}(s) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -s^{-1} & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{7.31}$$

on the R-matrix of $\mathcal{U}(sl(2))$: $R = 1\!\!1_{4\times 4}$.

We find the universal form of the twist to be

$$\mathcal{F}(s) = \sum_{n=0}^{\infty} \frac{1}{n!} \left(\prod_{k=0}^{n-1} [(1+k-s)1 - h] \otimes 1 \right)^{-1} e^n \otimes f^n. \tag{7.32}$$

This twist is the scaling limit of the universal twist (7.30). We have checked that it satisfies the shifted cocycle condition (A.10). This promotes $\mathcal{U}_s(sl(2))$ to the level of a quasitriangular quasi-Hopf algebra (QTQHA) as a twist of $\mathcal{U}(sl(2))$ with universal $\mathcal{R}$-matrix $\mathcal{R}_{\mathcal{U}_s} = \mathcal{F}_{21}(s)\mathcal{F}_{12}^{-1}(s)$ and coproduct

$$\begin{aligned}
\Delta(h) &= h \otimes 1 + 1 \otimes h \;\equiv\; h_1 + h_2 \\
\Delta(e) &= 1 \otimes e + \frac{1 - s - h_1 - h_2}{1 - s - h_1} e \otimes 1 \cdot \phi(s)^{-1} \\
\Delta(f) &= \left[\phi(s) \cdot f \otimes 1 + \frac{s}{s + h_1} 1 \otimes f\right] \phi(s - 1)
\end{aligned} \tag{7.33}$$

where

$$\phi(s) = \Phi_{312}(s)\big|_{h_3=1} = \mathcal{F}_{12}(s)\mathcal{F}_{12}(s + h_3)^{-1}\big|_{h_3=1} = 1 \otimes 1 + \frac{1}{(s + h_1)(1 - s - h_1)} e \otimes f. \tag{7.34}$$

The coproducts (7.33) are obtained using $\Delta(L(s)) = L_2(s + h_1)L_1(s)\pi_3(\Phi_{312}(s))^{-1}$, with $L(s)$ given in (5.31) and the last equality in (7.34) comes from the evaluation of $\Delta(L_{11} = 1)$.

8. Conclusion

We have now constructed several R-matrix representations for algebraic structures, deduced from vertex or face elliptic quantum $sl(2)$ algebras by suitable limit procedures. We have shown that these structures exhibited associativity properties characterized by (dynamical) Yang–Baxter equations for their evaluated R-matrices. Finally, we have constructed a reciprocal set of twist-like transformations, acting on the evaluated R-matrices canonically as $R_{12}^T = T_{21} R_{12} T_{12}^{-1}$.

The next step is now to try to get explicit universal formulae for these R-matrices and twist operators. This, in turn, requires specifying the exact form under which individual generators are encapsulated in the Lax matrices, and thus obtain the full description of the associative algebras which we wish to study.

Let us immediately indicate that we need, in particular, to separate (as is explained in [28]) the two algebraic structures contained in the single R-matrix formulations labelled here as (deformed) (dynamical) double Yangians $\mathcal{DY}_{...}(sl(2))_c$. Expansion of the Lax matrix in terms of integer labelled generators will lead to the (deformed) (dynamical) versions of the genuine double Yangian [29–32]; expansion in terms of Fourier modes by a contour integral will lead to the 'scaled elliptic' algebras [4, 28] more correctly labelled $\mathcal{A}_{\hbar,\eta}(\widehat{sl(2)_c})$. Once this is done, we can then start to investigate the following issues:

- representations, vertex operators;
- Hopf or quasi-Hopf algebra structure, leading to:
- universal R-matrices and twists.

Concerning these last two points a number of already known explicit results lead us to draw reasonable conjectures on some of the newly discovered algebraic structures in our work.

8.1. Known universal R-matrices and twists

Universal R-matrices are known for $\mathcal{U}_q(\widehat{sl(2)_c})$ [42]; $\mathcal{A}_{q,p}(\widehat{sl(2)_c})$ [12] and $\mathcal{B}_{q,p,\lambda}(\widehat{sl(2)_c})$ [12, 18]; $\mathcal{B}_{q,\lambda}(sl(2))$ [40]. They are also known for the double Yangian $\mathcal{DY}(sl(N))_c$ [30] (proved for $N = 2$, conjectured for $N \geqslant 3$). Universal twists have been constructed in the finite-algebra case from $\mathcal{U}_q(sl(N))$ to $\mathcal{B}_{q,\lambda}(sl(N))$ [12, 22, 36]; and in the affine case from $\mathcal{U}_q(\widehat{sl(2)_c})$ to $\mathcal{A}_{q,p}(\widehat{sl(2)_c})$ [12] and $\mathcal{B}_{q,p,\lambda}(\widehat{sl(2)_c})$ [34, 35].

8.2. Conjectures

We therefore expect that universal R-matrices and twist operators may be obtained for the complete set of algebraic structures represented by figure 5 in the vertex case and figure 6 in the face case. The structures $\mathcal{D}Y_{...}(sl(2))_c$ are to be interpreted here as genuine, integer-labelled double Yangians. The explicit construction of universal objects in this frame seems achievable, starting from the universal $\mathcal{R}$-matrix in [29, 30]. This construction was recently completed in the case of deformed double Yangians $\mathcal{D}Y_r^{F,V6,V8}(sl(2))_c$ [43]. The problem of constructing universal objects associated with the continuous-labelled algebras of $\mathcal{A}_{\hbar,\eta}$-type is more delicate, since one needs, in particular, to contrive a direct universal connection between continuous-labelled generators in $\mathcal{A}_{\hbar,\eta}$ and discrete-labelled generators in $\mathcal{A}_{qp}$, or between $\mathcal{A}_{\hbar,0}$ and $\mathcal{U}_q(\widehat{sl(2)}_c)$.

8.3. The case of unitary matrices

We have also identified 'homothetical' twist-like connections between $1\!\!1_{4\times4}$, interpreted as the evaluated R-matrix $1\!\!1$ for the centrally extended algebra $\mathcal{U}(\widehat{sl(2)}_c)$, and unitary R-matrices realizing an *RLL*-structure 'proportional' to $\mathcal{U}_q(\widehat{sl(2)}_c)$. By a homothetical twist-like connection, we mean the existence of an invertible matrix $F(z)$ such that two R-matrices are connected by

$$\widetilde{R} = f(z, p, q) F_{21}(z^{-1}) R F_{12}(z)^{-1} \tag{8.1}$$

where $f(z, p, q)$ is a c-number function.

We have also computed homothetical TLAs between each double Yangian-like structure and its antecedent structures through the scaling procedure.

Interpretation of this *RLL*-structure remains obscure. The canonical construction of universal R-matrices for $\mathcal{U}_q(\widehat{sl(2)}_c)$ [42] and their subsequent evaluation [39] leaves open the possibility of an alternative construction of universal → evaluated R-matrix which lead to unitary (and crossing-symmetrical) R-matrices; it may arise either by dropping the triangularity requirement $\mathcal{R} \in \mathcal{B}_+ \otimes \mathcal{B}_- \subset \mathcal{U}_q(\widehat{sl(2)}_c) \otimes \mathcal{U}_q(\widehat{sl(2)}_c)$, or by relaxing analyticity constraints on the evaluated R-matrix.

8.4. The notion of dynamical elliptic algebra

Finally, let us briefly comment on the notion of 'dynamical' algebraic structure. This notion was applied throughout this paper to algebras incorporating an extra parameter λ belonging to the Cartan algebra, subsequently shifted along a general Cartan algebra direction. This shift is therefore retained in the Yang–Baxter equation for evaluated R-matrices of face type (but *not* of vertex type, for which the extra parameter is simply a c-number and the shift takes place along the central charge direction, set to zero in the evaluation representation†). A particular illustration of this fact arises in the case of classical and quantum R-matrix for Calogero–Moser models [21], where λ is identified with the momentum of the Calogero–Moser particles, hence the denomination 'dynamical' for the R-matrices. In the algebraic structures described here, however, λ is not yet promoted to the rôle of generator, hence this denomination is a slight abuse of the notation. There exists, however, at least one example of algebraic structure, $\mathcal{U}_{q,p}(\widehat{sl(2)}_c)$ [3, 20], where λ and its conjugate $\frac{\partial}{\partial\lambda}$ are 'added' to the algebra $\mathcal{B}_{q,p,\lambda}(\widehat{sl(2)}_c)$; however, $\mathcal{U}_{q,p}(\widehat{sl(2)}_c)$ is not a Hopf, even quasi-Hopf, algebra. We

† This fact was clarified to us by O Babelon.

expect therefore that similar genuinely dynamical algebraic structures may be associated in the same way to all 'dynamical' algebras described here, and may play an important rôle in solving the models where such algebras arise.

Acknowledgments

This work was supported in part by CNRS and EC network contract number FMRX-CT96-0012. MR was supported by an EPSRC research grant no GR/K 79437. We wish to thank O Babelon for enlightening discussions, S M Khoroshkin for fruitful explanations and P Sorba for his helpful and clarifying comments. JA wishes to thank the LAPTH for its kind hospitality.

Appendix A. General properties of R-matrices and twists

All evaluated R-matrices in this paper obey one of the following equations, implying the associativity of the exchange algebra.

- Yang–Baxter equation:

$$R_{12}(z)\, R_{13}(zz')\, R_{23}(z') = R_{23}(z')\, R_{13}(zz')\, R_{12}(z) \tag{A.1}$$

$$R_{12}(\beta)\, R_{13}(\beta+\beta')\, R_{23}(\beta') = R_{23}(\beta')\, R_{13}(\beta+\beta')\, R_{12}(\beta). \tag{A.2}$$

- Dynamical Yang–Baxter equation:

$$R_{12}(z, \lambda + h^{(3)})\, R_{13}(zz', \lambda)\, R_{23}(z', \lambda + h^{(1)}) = R_{23}(z', \lambda)\, R_{13}(zz', \lambda + h^{(2)})\, R_{12}(z, \lambda) \tag{A.3}$$

$$R_{12}(\beta, \lambda + h^{(3)})\, R_{13}(\beta+\beta', \lambda)\, R_{23}(\beta', \lambda + h^{(1)}) = R_{23}(\beta', \lambda)\, R_{13}(\beta+\beta', \lambda + h^{(2)})\, R_{12}(\beta, \lambda) \tag{A.4}$$

depending upon the multiplicative or additive nature of the spectral parameter.

Among the algebraic structures which we consider here, some are known to have quasitriangular Hopf algebra (QTHA) structure (for instance $\mathcal{U}_q(\widehat{sl(2)_c})$, $\mathcal{D}Y(sl(2))_c)$ [44], and others are QTQHA [33] (for instance $\mathcal{A}_{q,p}(\widehat{sl(2)_c})$, $\mathcal{B}_{q,p,\lambda}(\widehat{sl(2)_c})$).

Their universal R-matrices $\mathcal{R} \in \mathfrak{A} \otimes \mathfrak{A}$ obey the universal Yang–Baxter equation in the first case,

$$\mathcal{R}_{12}\mathcal{R}_{13}\mathcal{R}_{23} = \mathcal{R}_{23}\mathcal{R}_{13}\mathcal{R}_{12} \tag{A.5}$$

and a more complicated Yang–Baxter-type equation in the second case, involving a cocycle $\Phi \in \mathfrak{A} \otimes \mathfrak{A} \otimes \mathfrak{A}$:

$$\mathcal{R}_{12}\Phi_{312}\mathcal{R}_{13}\Phi_{132}^{-1}\mathcal{R}_{23}\Phi_{123} = \Phi_{321}\mathcal{R}_{23}\Phi_{231}^{-1}\mathcal{R}_{13}\Phi_{213}\mathcal{R}_{12}. \tag{A.6}$$

However, in all the cases which are considered here, the R-matrices, once evaluated, obey the Yang–Baxter or dynamical Yang–Baxter equation.

We now recall the following contingent properties of evaluated R-matrices.

- Unitarity:

$$\begin{aligned} &R_{12}(z)\, R_{21}(z^{-1}) = \mathbb{1} \\ &R_{12}(\beta)\, R_{21}(-\beta) = \mathbb{1}. \end{aligned} \tag{A.7}$$

- Crossing-symmetry:

$$\begin{aligned} \left(R_{12}(x)^{t_2}\right)^{-1} &= \left(R_{12}(q^2x)^{-1}\right)^{t_2} \\ \left(R_{12}(\beta)^{t_2}\right)^{-1} &= \left(R_{12}(\beta - 2\mathrm{i}\pi)^{-1}\right)^{t_2} \end{aligned} \tag{A.8}$$

depending upon the multiplicative or additive nature of the spectral parameter.

The unitarity relation is not satisfied in most cases: the already known evaluated R-matrices for $\mathcal{A}_{q,p}(\widehat{sl(2)}_c)$, $\mathcal{B}_{q,p,\lambda}(\widehat{sl(2)}_c)$, $\mathcal{U}_{q,\lambda}(\widehat{sl(2)}_c)$ only obey the crossing relation (A.8) [12, 39].

We have indicated that universal twist operators $\mathcal{F}$ transform a coproduct Δ into another one $\Delta^{\mathcal{F}}(\cdot) = \mathcal{F}\Delta(\cdot)\mathcal{F}^{-1}$ and the $\mathcal{R}$-matrix into $\mathcal{R}^{\mathcal{F}} = \mathcal{F}_{21}\mathcal{R}\mathcal{F}_{12}^{-1}$. If now $(\mathfrak{A}, \Delta, \mathcal{R})$ defines a quasi-triangular Hopf algebra and $\mathcal{F}$ satisfies the cocycle condition

$$\mathcal{F}_{12}(\Delta \otimes \mathrm{id})\mathcal{F} = \mathcal{F}_{23}(\mathrm{id} \otimes \Delta)\mathcal{F}. \tag{A.9}$$

$(\mathfrak{A}, \Delta^{\mathcal{F}}, \mathcal{R}^{\mathcal{F}})$ defines again a quasi-triangular Hopf algebra. If, however, $\mathcal{F}$ satisfies no particular cocycle-like relation, $(\mathfrak{A}, \Delta^{\mathcal{F}}, \mathcal{R}^{\mathcal{F}})$ defines a QTQHA: $\mathcal{R}^{\mathcal{F}}$ satisfies then the YB-type equation (A.6). An interesting intermediate structure arises when $\mathcal{F}$ satisfies a so-called shifted cocycle condition, depending upon a parameter λ such that [18, 22]:

$$\mathcal{F}_{12}(\lambda)(\Delta \otimes \mathrm{id})\mathcal{F} = \mathcal{F}_{23}(\lambda + h^{(1)})(\mathrm{id} \otimes \Delta)\mathcal{F} \tag{A.10}$$

where $h \in \mathfrak{A}$. In this case, $\mathcal{R}^{\mathcal{F}}$ satisfies the universal form of the dynamical Yang–Baxter equation (A.3).

Appendix B. Useful functions

The infinite multiple products are defined by

$$(z; p_1, \ldots, p_m)_\infty = \prod_{n_i \geqslant 0} (1 - z p_1^{n_1} \ldots p_m^{n_m}). \tag{B.11}$$

The Θ functions used in this paper are defined by

$$\Theta_a(x) = (x; a)_\infty (a x^{-1}; a)_\infty (a; a)_\infty \tag{B.12}$$

Γ_r is the multiple Gamma function of order r given by

$$\Gamma_r(x|\omega_1, \ldots, \omega_r) = \exp\left(\frac{\partial}{\partial s} \sum_{n_1,\ldots,n_r \geqslant 0} (x + n_1\omega_1 + \cdots + n_r\omega_r)^{-s}\bigg|_{s=0}\right). \tag{B.13}$$

In particular,

$$\Gamma_1(x|\omega_1) = \frac{\omega_1^{x/\omega_1}}{\sqrt{2\pi\omega_1}} \Gamma\left(\frac{x}{\omega_1}\right).$$

It has the following property:

$$\frac{\Gamma_r(x + \omega_i|\omega_1, \ldots \omega_r)}{\Gamma_r(x|\omega_1, \ldots \omega_r)} = \frac{1}{\Gamma_{r-1}(x|\omega_1, \ldots \omega_{i-1}, \omega_{i+1} \ldots, \omega_r)}. \tag{B.14}$$

Multiple sine functions are defined by

$$S_r(x|\omega_1, \ldots \omega_r) = \Gamma_r(x|\omega_1, \ldots \omega_r)^{-1} \Gamma_r(\omega_1 + \cdots + \omega_r - x|\omega_1, \ldots \omega_r)^{(-1)^r}. \tag{B.15}$$

In particular,

$$S_1(x|\omega_1) = 2\sin\left(\frac{\pi x}{\omega_1}\right)$$

and the Barnes double sine function of periods ω_1 and ω_2 is given by [45]

$$S_2(x|\omega_1, \omega_2) = \frac{\Gamma_2(\omega_1 + \omega_2 - x|\omega_1, \omega_2)}{\Gamma_2(x|\omega_1, \omega_2)}. \tag{B.16}$$

The q-hypergeometric function

$$ {}_2\phi_1\left(\begin{matrix} q^a & q^b \\ & q^c \end{matrix} ; q, z\right)$$

is defined by

$$ {}_2\phi_1\left(\begin{matrix} q^a & q^b \\ & q^c \end{matrix} ; q, z\right) = \sum_{n=0}^{\infty} \frac{(q^a;q)_n (q^b;q)_n}{(q^c;q)_n (q;q)_n} z^n \qquad \text{where} \quad (x;q)_n = \prod_{k=0}^{n-1}(1 - xq^k). \tag{B.17}$$

References

[1] Jimbo M, Kedem R, Konno H, Miwa T and Weston R 1995 Difference equations in spin chains with a boundary *Nucl. Phys.* B **448** 429
(Jimbo M, Kedem R, Konno H, Miwa T and Weston R 1995 *Preprint* hep-th/9502060)

[2] Andrews G, Baxter R and Forrester P J 1984 Eight-vertex SOS model and generalized Rogers-Ramanujan type identities *J. Stat. Phys.* **35** 193

[3] Konno H 1998 An elliptic algebra $U_{q,p}(\widehat{sl}(2))$ and the fusion RSOS model *Commun. Math. Phys.* **195** 373

[4] Konno H 2000 Degeneration of the elliptic algebra $\mathcal{A}_{q,p}(\widehat{sl}(2))$ and form factors in the sine–Gordon theory *Proc. Nankai–CRM Joint Meeting on Extended and Quantum Algebras and their Applications to Physics (Tianjin, 1996) (CRM Series in Mathematical Physics)* (Berlin: Springer) to appear
(Konno H 1997 *Preprint* hep-th/9701034)

[5] Khoroshkin S, LeClair A and Pakuliak S 1999 Angular quantization of the Sine–Gordon model at the free fermion point *Preprint* hep-th/9904082

[6] Baxter R J 1972 Partition function of the eight-vertex lattice model *Ann. Phys.* **70** 193

[7] BelavinA A 1981 Dynamical symmetry of integrable quantum systems *Nucl. Phys.* B **180** 189

[8] Date E, Jimbo M, Miwa T and Okado M 1986 Fusion of the eight-vertex SOS model *Lett. Math. Phys.* **12** 209

[9] Jimbo M, Miwa T and Okado M 1987 *Lett. Math. Phys.* **14** 123
Jimbo M, Miwa T and Okado M 1988 Solvable lattice models related to the vector representation of classical simple Lie algebras *Commun. Math. Phys.* **116** 507

[10] Sklyanin E K 1982 Some algebraic structures connected with the Yang–Baxter equation *Funct. Anal. Appl.* **16** 263
Sklyanin E K 1983 Some algebraic structures connected with the Yang–Baxter equation. Representations of quantum algebras *Funct. Anal. Appl.* **17** 273

[11] Foda O, Iohara K, Jimbo M, Kedem R, Miwa T and Yan H 1994 An elliptic quantum algebra for $\widehat{sl}_2$ *Lett. Math. Phys.* **32** 259
(Foda O, Iohara K, Jimbo M, Kedem R, Miwa T and Yan H 1994 *Preprint* hep-th/9403094)

[12] Jimbo M, Konno H, Odake S and Shiraishi J 2000 Quasi-Hopf twistors for elliptic quantum groups *Transform. Groups* to appear
(Jimbo M, Konno H, Odake S and Shiraishi J 1997 *Preprint* q-alg/9712029)

[13] Awata H, Kubo H, Odake S and Shiraishi J 1996 Quantum $\mathcal{W}_N$ algebras and Macdonald polynomials *Commun. Math. Phys.* **179** 401
(Awata H, Kubo H, Odake S and Shiraishi J 1995 *Preprint* q-alg/9508011)
Shiraishi J, Kubo H, Awata H and S Odake A 1996 Quantum deformation of the virasoro algebra and the Macdonald symmetric functions *Lett. Math. Phys.* **38** 33
(Shiraishi J, Kubo H, Awata H and S Odake A 1995 *Preprint* q-alg/9507034)

[14] Frenkel E and Reshetikhin N 1996 Quantum affine algebras and deformations of the Virasoro and W-algebras *Commun. Math. Phys.* **178** 237
(Frenkel E and Reshetikhin N 1995 *Preprint* q-alg/9505025)

[15] Feigin B and Frenkel E 1996 Quantum W algebras and elliptic algebras *Commun. Math. Phys.* **178** 653
(Feigin B and Frenkel E 1995 *Preprint* q-alg/9508009)

[16] Avan J, Frappat L, Rossi M and Sorba P 1999 Deformed $\mathcal{W}_N$ algebras from elliptic $sl(N)$ algebras *Commun. Math. Phys.* **199** 697
(Avan J, Frappat L, Rossi M and Sorba P 1998 *Preprint* math QA/9801105)

[17] Avan J, Frappat L, Rossi M and Sorba P 1999 Universal construction of q-deformed $\mathcal{W}$ algebras *Commun. Math. Phys.* **202** 445
(Avan J, Frappat L, Rossi M and Sorba P 1998 *Preprint* math QA/9807048)

[18] Felder G 1994 Elliptic quantum groups *Proc. ICMP Paris* p 211
See also Gervais J L and Neveu A 1984 Novel triangle relation and absence of tachyons in Liouville string field theory *Nucl. Phys.* B **238** 125

[19] Enriquez B and Felder G 1997 Elliptic quantum groups $E_{\tau\eta}(sl_2)$ and quasi-Hopf algebras *Preprint* q-alg/9703018

[20] Jimbo M, Konno H, Odake S and Shiraishi J 1999 Elliptic algebra $U_{q,p}(\widehat{sl}_2)$: Drinfel'd currents and vertex operators *Commun. Math. Phys.* **199** 605
(Jimbo M, Konno H, Odake S and Shiraishi J 1998 *Preprint* math. QA/9802002)

[21] Avan J, Babelon O and Billey E 1996 The Gervais–Neveu–Felder equation and quantum Calogero–Moser systems *Commun. Math. Phys.* **178** 281

[22] Babelon O, Bernard D, Billey E 1996 A quasi-Hopf algebra interpretation of quantum 3-j and 6-j symbols and difference equations *Phys. Lett.* B **375** 89

[23] Jurčo B, Schupp P 1997 AKS scheme for face and Calogero–Moser–Sutherland type models *Preprint* solv-int/9710006

[24] Hou B Y and Yang W L 1997 Dynamically twisted algebra $\mathcal{A}_{q,p;\pi}(\widehat{gl_2})$ as current algebra generalizing screening currents of q deformed Virasoro algebra *Preprint* q-alg/9709024

[25] Reshetikhin N Yu, Semenov-Tian-Shansky M A 1990 Central extensions of quantum current groups *Lett. Math. Phys.* **19** 133

[26] Ding J, Frenkel I B 1993 Isomorphism of two realizations of quantum affine algebra $U_q(\widehat{gl}(n))$ *Commun. Math. Phys.* **156** 277

[27] Jimbo M, Konno H and Miwa T 1996 Massless *XXZ* model and degeneration of the elliptic algebra $\mathcal{A}_{q,p}(\widehat{sl}(2))$ *Deformation theory and Symplectic Geometry (Ascona)* pp 117–38
(Jimbo M, Konno H and Miwa T 1996 *Preprint* hep-th/9610079)

[28] Khoroshkin S, Lebedev D and Pakuliak S 1998 Elliptic algebra $\mathcal{A}_{q,p}(s\hat{l}_2)$ in the scaling limit *Commun. Math. Phys.* **190** 597
(Khoroshkin S, Lebedev D and Pakuliak S 1997 *Preprint* q-alg/9702002)

[29] Khoroshkin S M and Tolstoy V N 1994 Yangian double and rational R matrix *Preprint* hep-th/9406194

[30] Khoroshkin S M 1996 Central extension of the Yangian double *Collection SMF, 7ème Rencontre du Contact Franco–Belge en Algèbre (Reims 1995) Preprint* q-alg/9602031

[31] Iohara K and Kohno M 1996 A central extension of $\mathcal{D}Y_\hbar(\widehat{gl}(2))$ and its vertex representations *Lett. Math. Phys.* **37** 319
(Iohara K and Kohno M 1996 *Preprint* q-alg/9603032)

[32] Iohara K 1996 Bosonic representations of Yangian double $\mathcal{D}Y_\hbar(\widehat{gl})$ with $\widehat{gl} = \widehat{gl}_N \widehat{sl}_N$ *J. Phys. A: Math. Gen.* **29** 4593
(Iohara K 1996 *Preprint* q-alg/9603033)

[33] Drinfel'd V G 1990 Quasi-Hopf algebras *Leningrad Math. J.* **1** 1419

[34] Frønsdal C 1997 Quasi Hopf deformations of quantum groups *Lett Math. Phys.* **40** 117
(Frønsdal C 1996 *Preprint* q-alg/9611028)

[35] Frønsdal C 1997 Generalization and exact deformations of quantum groups *Publ. RIMS Kyoto Univ.* **33** 91

[36] Arnaudon D, Buffenoir E, Ragoucy E and Roche Ph 1998 Universal solutions of quantum dynamical Yang–Baxter equations *Lett. Math. Phys.* **44** 201–14
(Arnaudon D, Buffenoir E, Ragoucy E and Roche Ph 1997 *Preprint* q-alg/9712037)

[37] Buffenoir E and Roche Ph 1999 Harmonic analysis on the quantum Lorentz group *Commun. Math. Phys.* **207** 499–555
(Buffenoir E and Roche Ph 1997 *Preprint* q-alg/9710022)

[38] Faddeev L D, Reshetikhin N Yu and Takhtajan L A 1990 Quantization of Lie groups and Lie algebras *Leningrad Math. J.* **1** 193

[39] Idzumi M, Iohara K, Jimbo M, Miwa T, Nakashima T and Tokihiro T 1993 Quantum affine symmetry in vertex

models *Int. J. Mod. Phys.* A **8** 1479
(Idzumi M, Iohara K, Jimbo M, Miwa T, Nakashima T and Tokihiro T 1992 *Preprint* hep-th/9208066)
[40] Babelon O 1991 Universal exchange algebra for Bloch waves and Liouville theory *Commun. Math. Phys.* **139** 619
[41] Arnaudon D, Avan J, Frappat L and Rossi M 2000 Deformed double Yangian structures *Rev. Math. Phys.* **12** at press
(Arnaudon D, Avan J, Frappat L and Rossi M 1999 *Preprint* math. QA/9905100)
[42] Khoroshkin S and Tolstoy V 1992 Uniqueness theorem for universal R-matrix *Lett. Math. Phys.* **24** 231
[43] Arnaudon D, Avan J, Frappat L, Ragoucy É and Rossi M 2000 On the quasi-Hopf structure of deformed double Yangians *Lett. Math. Phys.* **51** 193
(Arnaudon D, Avan J, Frappat L, Ragoucy É and Rossi M 2000 *Preprint* math. QA/0001034)
[44] Drinfeld V G 1986 Quantum Groups *Proc. Int. Congress of Mathematicians (Berkeley, CA)* vol 1 (New York: Academic) p 798
[45] Barnes E W 1901 The theory of the double gamma function *Phil. Trans. R. Soc.* A **196** 265–388